# Making Functional Mathematics Happen

*Comments to QCA and DfES*
*from*
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## *Our background in FM*

The Shell Centre team is a group of 'tool design engineers'[1] which uses an engineering research approach to develop new or improved tools to help practitioners achieve targets for improvement in education, particularly in Mathematics and its uses in real world problem solving (FM).

This approach uses methodology that is standard in, say, engineering or medicine. It uses:

- prior insight-focused research in guiding design,
- design skills
- research methods of systematic refinement of products through feedback from successive rounds of trials.

The team is also part of the international *Mathematics Assessment Resource Service* (MARS) and has designed assessment for many school systems worldwide, including QCA and UK examination bodies.

Earlier developments in the area of Functional Mathematics include *The Language of Functions and Graphs* (Swan et al 1986) and *Numeracy through Problem Solving* (Shell Centre 1987-89) both developed with the then Joint Matriculation Board.

UK interest in FM declined through the implementation, though not the original design, of the National Curriculum (see below). We welcome the spirit of the current initiative and greatly hope, but do not expect, that it will be realised in practice.

[1] Other, larger teams with this approach include those at the Freudenthal Institute in the Netherlands and, in the US, EDC, TERC, COMAP, and the Lawrence Hall of Science.

# Making Functional Mathematics Happen

## Executive Summary

These comments are in response to the following recent QCA documents:

*Functional skill subject definitions* and *On Developing Functional Skills Qualifications;*

*Functional Mathematics Standards*, along with those for ICT and English;

Ken Boston's recent speech on Mathematics to ACME.

We will focus on Functional Mathematics (FM), with which we have been creatively involved for more than two decades of research, development and classroom practice.

**Strengths** that seem likely to forward the functionality of school mathematics include:

*Functional skills subject definitions* sets out the principles admirably;

*Level descriptions* emphasise the *processes* of using mathematics for real life problems;

Recognition that 'potentially useful' mathematics (all of it) is not, on its own, functional.

**Concerns** The current proposals also have weaknesses that, as they stand, make it very likely that *the outcome will be non-functional mathematics for most students*, and 'business pretty much as usual' in mathematics classrooms, and tests. Why do we make this assertion? The reasons are outlined in this paper. In summary:

**The FM curriculum is currently underspecified** allowing interpretations close to the current status quo. To avoid this, *Performance targets* need to be specified, and *exemplified*.

**The assessment tasks will determine the implemented curriculum**. The current draft specification, if used as the basis for commissioning test design, allows a wide range of realizations; most would, as now, not assess functional mathematics. To avoid this, high-stakes tests need to be reasonably *balanced* across the performance goals of the curriculum.

**Functional mathematics involves non-routine problems**, ones you have not been shown exactly how to solve. In such problems you have to *find*, not just to try to *remember*, solution paths, using your mathematical toolkit, large or small. Common in other subjects, this kind of performance is not currently tested, or taught, in Mathematics. It can be, as it has been in the past.

**The progression between levels is specified entirely in terms of content**, the list of mathematical topics to be covered. That is not the only dimension in which real problem solving progresses; it is equally characterised by *more sophisticated use*, on *more complex problems*, of relatively simple mathematics. To make this clear in the level descriptions, substantial *exemplification* is needed since problem-solving processes are much the same at all levels.

**Realising functional mathematics is a major challenge** Is that fully recognized? Absorbing and implementing these changes will take time and support. Teachers will need well-engineered teaching materials (cheap), with some professional development support (expensive). Assessment providers will need help with the design and development of non-routine assessment tasks for Functional Mathematics; they have neither recent experience nor adequate methodology. The overall 'change model' must be powerful and robust, being designed to learn as it proceeds.

While we are sure that QCA and DfES already plan to address most of these areas in future, we believe a full analysis and detailed planning is essential at this stage. It is well-known, in general, that tackling complex problems piecemeal guarantees inferior outcomes. High-quality examples of solutions to all these problems have been developed in the past; they should be evaluated in depth, with further work building on them.

*Contents*

## Introduction

In this paper we focus on Functional Mathematics (FM), with which we have been creatively involved for more than two decades of research, development and classroom practice. We will bear in mind what we have learnt at the QCA-ACME consultative meetings in March and June of 2005. We believe that what we have to say is in harmony with the, to us rather surprising, consensus among speakers at those meetings, and with the Smith and Tomlinson Reports.

**Strengths of the current proposals**
The QCA documents released so far show real strengths, notably in the:

- general review of the FM situation in Ken Boston's speech to ACME;
- *Functional skills subject definitions*;
- emphasis in the *Functional Mathematics Standards*, as the first section of each Level description, on the *processes* of tackling real situations, which seem well described.
- recognition that 'potentially useful' mathematics (all of it) is not, on its own, functional

The 'Functional skills subject definitions' set out the guiding principles for Functional Mathematics, notably (our italics):

> "Functional skills are those core elements of English, mathematics and ICT that provide an individual with the essential knowledge, skills and understanding that will enable them to operate confidently, effectively and independently in life and at work.
>
> .....
>
> Each individual has sufficient understanding of a range of mathematical concepts and is able to know how and when to use them. For example, *they will have the confidence and capability to use maths to solve problems embedded in increasingly complex settings and to use a range of tools, including ICT as appropriate.*
>
> .....
>
> In life and work, each individual will develop the analytical and reasoning skills to *draw conclusions, justify how they are reached and identify errors or inconsistencies. They will also be able to validate and interpret results, to judge the limits of their validity and use them effectively and efficiently.*"

This is all great stuff with which few, we hope, will disagree. However, as Ken Boston notes in his speech, it has not proved easy to realize Functional Mathematics in most classrooms around the world [2]. Hence the analysis below is focused on the challenges of making these principles a classroom reality in typical classrooms. (The good news is that there exist exemplars that show that it can be done in realistic circumstances of personnel and support, e.g Shell Centre 1987-89talics above highlight the areas that will present the greatest challenge – areas where current curricula in Mathematics contribute little or nothing. We shall return to these principles in discussing mechanisms for turning these principles into classroom practice.

**Functional Mathematics for all abilities** We have one caveat. The impression is often given that functionality with mathematics is a problem only for low achieving students; in fact, it affects at all levels of ability. It is fair to assert that

**Currently, for most people, their secondary mathematics is non-functional.**

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[2] Like him, we do not think it useful to distinguish here between *functional mathematics, mathematical literacy, quantitative literacy* and *numeracy* (in its original, Crowther Report sense,as "the mathematical equivalent of literacy") Internationally, the words are seen as roughly equivalent.

If you doubt this, ask some adults who work in occupations that do not require specialist mathematical skills when they last used any mathematics that they were first taught in secondary school.

More systematically, ask them to tackle the tasks in Paper A in the Appendix. We believe these tasks should be straightforward for all well-educated adults (for example, graduates in Humanities subjects, or Law!); currently, as the notes accompanying the tests describe, it is not. (Paper B is similarly designed to illustrate FM at GCSE level). These papers surely make clear that Functional Mathematics can be challenging up to a high level. (Indeed, some inward-looking mathematicians argue that functionality is too difficult to form a reasonable target; successful practical implementations have shown that is not so.)

Over 1000 hours of school time are spent on secondary mathematics up to GCSE – several times that allocated for other culturally important areas such as music. Is Government is happy with its non-functionality, except for future specialists? It seems not to satisfy the QCA principle that:

> *Functional skills …… will enable them to operate confidently, effectively and independently in life and at work.*

After 1000 hours of study, most of their secondary mathematics should surely be functional for all students.

Experience suggests that this does *not* mean substantially less content, though content and teaching priorities will be somewhat different. FM will produce much higher competence with that content.

The discussion in this paper therefore applies to FM across the ability range, and for groups within it.

**Concerns**

Complementing their strengths, the current proposals also have serious weaknesses that, as they stand, are likely to prevent the realization of these principles. In this paper we explain how and why, and recommend ways in which the outcomes can be brought closer to the unimpeachable intentions.

The concerns are discussed under the following headings:

- The FM curriculum is currently underspecified
- The assessment tasks will determine the implemented curriculum
- Functional mathematics involves non-routine problems
- Progression between levels is multi-dimensional
- The implications for teaching are profound
- Realising functional mathematics is a major challenge

We are confident that you have most of these things in mind. However, experience suggests that, unless these elements of a successful path of change are more fully specified and understood at this stage, major distortion of the excellent principles set out by QCA will occur in implementation – usually through the unintended consequences of apparently unconnected decisions[3].

**Lessons from English**

We believe, from our own and others' experience, that Mathematics has much to learn from those who are concerned with curriculum and assessment in English, where functionality has long been an accepted responsibility, not just in principle but in practice. English teachers regularly bring students' life experience into their teaching, and give students tasks that are directly relevant to life goals. Further, they find this a

[3] For example, the FM Standards as presented may be seen, as they were with National Curriculum assessment, to allow test developers to offer minor modifications of current tests, simply adding a veneer of practical context to short items.

help in advancing more academic goals. We shall return to this analogy from time to time.

**This paper**

You will find this paper far from polished. It has been prepared rapidly so as to respond to QCAs documents within the period allowed. However, it reflects four decades of experience, research and development, by ourselves and others, on the teaching of functional mathematics. Our goal here is to set down the main points that we believe must be taken into account by Government, if functional mathematics is to have a good chance of becoming a reality in England and Wales.

We will welcome comments – please send them to *Hugh.Burkhardt@nottingham.ac.uk*.
If invited, we shall be happy to refine the analysis further.
Further, we are keen to take part in the kinds of development we recommend.

Some of the analysis that follows is complex and technical, as it must be for a complex problem, but we have tried to make it accessible to the non-specialist reader.

## The FM curriculum is currently underspecified

The targets in the current draft are underspecified, allowing interpretations close to the current status quo. As so often with detailed analytic *attainment targets*, it is impossible to find the *performance targets*, what students will actually be expected to do -- surely a key aspect of 'standards'. For example, these Standards could be assessed entirely through short fragmentary tasks or subtasks (as with present Mathematics tests), or through 10 minute holistic examination questions, or through substantial autonomous student coursework, or any balance of task-types within this range. Each of these would represent quite different *performance targets* and require a different implemented curriculum in the classroom.

**Specifying a curriculum**

It is not straightforward to find a good model for specifying a new curriculum, or a curriculum element like FM. By a good model, we mean one where the intended outcomes in terms of classroom learning activities and student performance are specified sufficiently unambiguously so that, provided the specification is honestly followed, the implemented outcomes are in line with the intentions.

This is not the place for a broad discussion of this issue; rather we shall focus on the recent history of a model that has lessons for the current initiative – the National Curriculum, with its three original aspects: Attainment Targets, Programmes of Study, and Assessment and Testing. In an earlier paper[4] we had suggested that these could form a good model, *provided each was specified separately and exemplified in appropriate detail*. In this view:

- **Attainment Targets** provide an analytic description of the expected domain of knowledge, including the *elements* of performance, at an appropriate level of detail.
- **Assessment and Testing**, in describing how the subject will be assessed, also provides through task exemplars a *holistic* picture of the expected range and variety of performance in the subject, an essential complement to the *analytic* scheme in the attainment targets.
- **Programmes of Study** describe the range and balance of classroom learning activities that are needed to develop the knowledge and performance.

[4] Burkhardt, H. 1987, *On Specifying a National Curriculum in Developments in School Mathematics Worldwide*, ed. I Wirszup, University of Chicago School Mathematics Project.

**Unintended distortions – a recent case**

These dimensions had also been implicitly recognised earlier, and described as essential in the Cockcroft Report(e.g. Para 243) and the National Curriculum Programmes of Study. However, the management of National Curriculum Mathematics Working Group, particularly the senior civil servants steering the group, saw a quite different relationship between the three, with *Programmes of Study* and *Assessment and Testing* as derivative, following from the *Attainment Targets* without further specification. "The Attainment Targets specify what they need to know, the Programmes of Study teach it, and the Assessment tests how much of it they know" is close to a direct quote.

This is nonsense – in mathematics, as in most kinds of performance, the whole is more than the sum of the parts. The attainment targets describe *elements* of performance that could either be tested separately as short items, or as part of large integrated projects, or by many types of task in between. These constitute very different kinds of performance in the subject – as different as a spelling test and an extended essay in English.

However, the Mathematics Working Group's management were not mathematically sophisticated. They drove the group to focus on specification of the Attainment Targets through detailed, "unambiguous" *Statements of Attainment* ("Things that a pupil can or can not do") and the testing was required to test these separately – an extremely fragmented definition of performance in Mathematics that excludes the longer chains of reasoning that Functional Mathematics requires.

The *Statements of Attainment* were soon recognised as a naive view of criterion-referenced assessment and abolished a few years later; however, by then the pattern of fragmentary item design was established in Key Stage tests and GCSE. It remains to this day – the average *reasoning length*[5] is about 90 seconds; *thinking with mathematics*, like writing essays, is not like that.

The decision pathway of this distortion is a case study that is highly relevant now. The Mathematics Working Group, as well as specifying the Statements of Attainment, included in its report a 40-page appendix of assessment tasks, illustrating the range of task types that should be used in assessing National Curriculum Mathematics. These can be found in *Mathematics for ages 5 to 16: Proposals of the Secretary of State for Education and Science and the Secretary of State for Wales,* issued for consultation in August 1988.

These tasks had mainly been developed with examining boards in the 1980s. They contain a lot of Functional Mathematics.

In a paper in preparation, we will illustrate, and analyse in some detail, the differences between those tasks and current Key Stage and GCSE assessment; they cover much the same mathematical *content* but are otherwise fundamentally different.

Here we make the point through just one substantial task from the Report , shown opposite– it is fairly typical, though the range there is wide, as it should be.

[5] *Reasoning length* is defined as the time a typical student needs to complete a prompted piece of a question (see the Framework for Balance, below)

The following task (in our view, still far from ideal) is from a 1980s GCSE.

(*The map is not shown here*)

> BRIDGES 1 hour
>
> *Remember: Show all your working clearly, so that someone else can follow what you did.*
>
> The scale of the map is 2cm to 1km.
>
> F and M are two towns.
>
> F is built on firm ground. M is built on marshy ground.
>
> The boundary line between the firm ground and the marshy ground is shown on the map, and is crossed by the River Am.
>
> The two towns F and M are to be joined by a road, made up on ONLY one or two straight line sections.
>
> Part of the road will cross firm ground and part will cross marshy ground.
>
> On the firm ground it costs £1 million to build 1km of road.
>
> On the marshy ground it costs £3 million to build 1km of road.
>
> Each time the road crosses the river it costs £2.5 million for a bridge.
>
> One possible route for the road is shown by the dotted lines.
>
> a) Find the length in km of each part, i.e. the firm part and the marshy part, of the dotted route. (Include the width of the river in your measurements).
>
> Calculate the cost of building each part of the road, the cost of bridges, and the total cost of the whole road.
>
> b) The dotted route is not the cheapest road which can be built.
>
> Try out as many different types of routes as you can, and work out the total cost of building each one.
>
> Label each different route on your map with a letter. Your aim is to find the cheapest possible route.

Compare this with the questions on similar content, overleaf, from recent GCSE papers.

Question1 is on using a scale. Question 2 is on calculating costings.

These two types of task assess quite different kinds of performance, representing a different view of what 'doing mathematics' means. There is no doubt as to which is more functional.

In the recent questions, the situation is not taken seriously – it is simply a context for a routine 'word problem'. The 'reasoning lengths' for each part are a minute or two, at most, involving just a few operations. The contrast with *Bridges* is stark.

You will find that the tasks in the appendix to the Secretary of State's 1988 National Curriculum proposals, mostly used previously by examining boards, contain a lot of functional mathematics.

1. (*again the outline map, with A and B marked, is not reproduced here*)

The map of this island is drawn to a scale of 3cm to represent 1km.
The port is at A and the airport is at B.

- Use the map to find the distance *AB* in kilometres.

  Answer________km

- The Lion Hotel is 2km from *A* on a bearing of 150°.
  Use a cross to mark the position of the hotel on the map.

2.

**Northern Rental**

**Van Hire Charges**

£24 per day
plus
12 pence per mile

Anita hires a van for one day.
She drives 68 miles.

How much is the total hire charge?

Answer £_______

John hires a van for two days.
The total hire charge is £66.72.

How many miles did he drive?

Answer _______miles

The implementation route, from one type of task to another that is so different, shows how detailed technical decisions can, and often do, have predictable unintended consequences. To avoid this, the overall goals have to be kept constantly in focus. In this case, the task exemplars in the August 1988 proposals were not re-considered and rejected; the appendix of tasks was simply removed from subsequent National

Curriculum documents on the stated grounds that "Assessment is for TGAT[6]" – as though there are no subject specific elements in assessment. Of course, complex tasks like those in the Appendix did not fit the management view that the Statements of Attainment *are* Mathematics – a checklist of things you could, or could not, do. (They took a very different view in English, a subject that they understood better, which remains partly functional. Mathematics has much to learn from English in this and other areas) The assessment designers then designed tests to assess, not mathematics, but the Statements of Attainment in this simplistic model.

There are no villains here; put simply, a desirable goal – neat, simple assessment – was pursued in apparent ignorance of its inevitable but unintended consequence – fragmented, non-functional mathematics (indeed a travesty of mathematics itself)

Other mistakes were made – notably treating "Using and Applying" as if it were another area of mathematical content, parallel to Number, Algebra, Geometry and Data, rather than a separate *process* dimension of performance[7] that applies in all content areas. However, it was the absence of an assessment specification, with an exemplar task set that was most crucial in allowing distortion of the intentions.

**Current dangers**

We predict that the current specification will lead assessment providers to:

- design tests made of short items,
- trial them with students who have not been taught functional mathematics, who will perform poorly,
- make the items even simpler, because "Functional Mathematics" is a hurdle that most students must be able to pass,
- thus continue with a non-functional mathematics curriculum in schools

– unless the kind of steps recommended in this paper are taken.

**Recommended actions:** In the top-level official specification of Functional Mathematics, **describe** and **exemplify** all of the following for every level:

- the elements of knowledge, skill and strategy;
- the range and balance of performance targets, exemplified by actual assessment tasks and mark schemes covering the intended variety;
- the range and balance of learning activities, exemplified by lesson descriptions, supported by video[8] covering the intended variety.

Initial commissioning of exemplars for QCA to select from will aid realisation of the principles. Further work on the development of assessment and curriculum support can then be commissioned with confidence that it will reflect intentions.

## The assessment tasks will determine the implemented curriculum

We have asserted that, if functional mathematics is to become a reality, the description and exemplification in the top-level official specification is essential. We now explain why. If, in this section, we seem to belabour the point, it is because making the assessment match the goals, *in coverage not just in correlation*, is a necessary condition for success in this initiative.

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[6] *Task Group on Assessment and Testing*, which developed the overall assessment framework, including the Level structure.

[7] If the same approach had been adopted for English, *Listening, Speaking, Reading* and *Writing* would have been rolled into one, *Using and applying,* with *Spelling, Grammar, Syntax* and *Figures of speech* as the other four Attainment Targets. That policy makers understand English better than Mathematics partly reflects their own school experiences.

[8] The written word is not effective for communicating changed patterns of interaction between people – in the classroom or elsewhere. (Try describing a TV chat show to an 'alien')

**What You Test Is What You Get**
Policy makers are often reluctant, despite the evidence (eg Clarke and Stephens 2000), to accept that *What You Test Is What You Get.* (Teachers have never had any doubt that WYTIWYG is a fact of life – high-stakes testing is, after all, society's main measure of their competence).

If decision makers have any doubt about this, further evidence could easily be gathered.

> Equip OFSTED inspectors with a checklist of those aspects of performance that are currently tested, and of those in the National Curriculum that are not, and ask them to note what they observe in a sample of Mathematics lessons. For example:
>
> *How many of the subtask prompts in this lesson took typical successful students:*
> *less than one minute; 1-3 mins; 3-6 mins; more than 6 mins.*
>
> (Ask them to collect the task prompts concerned)
>
> Enough data to give a fair picture could be collected in a month or two and analysed in less. (We would be happy to draft such a protocol for consideration)

The reluctance of policy makers to accept WYTIWYG may be due to the responsibility it implies for designing **balanced assessment**, **designed so that "teaching to the test" requires an *implemented curriculum* in the classroom that is reasonably balanced across *all* the goals of the *intended curriculum*.**

**Simple tests – attractions and limitations**
"Simple tests" have great attractions:

- *for assessment providers* because they are easy to design and replicate – further, their "short independent items" validate the otherwise-unreasonable assumptions of traditional psychometrics, which is statistically sophisticated but in other respects naive. (Modern psychometrics has moved to recognise this, and copes better with complex tasks);
- *for teachers* because, for various reasons, they dislike all kinds of external testing – further, teaching for routine imitative tasks is easier;
- *for policy makers* because they are easy to understand, and cheap.

However there is a big price – such tests inevitably lead to simplistic, narrow implemented curricula in the classroom, and to non-functional outcomes.

Who would now suggest testing performance in:

- *Decathlon* by testing, on grounds of economy and correlation, only the 100 meters?
- *Piano* by testing only scales and arpeggios, with no complete pieces of music?
- *English* by testing only spelling, grammar and syntax, with no essays, stories,..?
- *Modern Languages* by testing grammar and not communication?
- …… and so on.

Yet mathematics is now tested in an comparably unbalanced way, all technique and no strategy or tactics. It is no wonder that many adults, at all levels of ability, find their school mathematics entirely non-functional – they have never been taught to use it effectively.

For functional mathematics, let us return to the principles just emphasised by QCA.[9]

> "....they will have the confidence and capability to use maths to solve problems embedded in increasingly complex settings and to use a range of tools, including ICT as appropriate.
>
> "....each individual will develop the analytical and reasoning skills to draw conclusions, justify how they are reached and identify errors or inconsistencies.

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[9] In fact, the National Curriculum for Mathematics as a whole specifies a much broader range of performance than is assessed in current high-stakes tests – notably non-routine problem solving, more open investigation and the longer chains of reasoning that doing mathematics really involves. It is the form and interpretation of the statutory elements that undermined this.

"They will also be able to validate and interpret results, to judge the limits of their validity and use them effectively and efficiently."

None of these are addressed in current Key Stage and GCSE tests, which only assess:

*short chains of reasoning* (average 90 seconds per prompted subtask)

*routine tasks*, very like those students have learnt and practised many times

*'practical' contexts that are stylized*, with no questions on assumptions, practical implications, reasonableness,…..

.... the list goes on.

As the QCA principles make clear, functional mathematics is not like that. Real life often involves problems that are new to the solver, take time and thought (not just memory) to analyse, and involve serious thinking about the practical situation. These things can be assessed and have been assessed, at various times in various ways over the last forty years, in high-stakes UK examinations for this age range[10] (see e.g ICTMA 1981–, Blum et al 2006)

Piaget noted that all significant modes of thought have successive levels of understanding:

- **imitation**
- **retention**
- **explanation**
- **adaptation**
- **extension**

Traditional, and our current, mathematics curricula address only the first two; functional mathematics requires all five.

This does not imply that it has to be more difficult, though at high levels it can be. Lynn Steen has noted that:

*Whereas school mathematics stresses elementary uses of sophisticated mathematics, mathematical literacy focuses on sophisticated uses of (often) elementary mathematics.*

A great deal can be done with simple mathematics, if you learn how. Many children come to school at age 5, already highly functional with counting. Curricula need to build on that.

Steen also remarked that functional mathematics is more like essay writing than mathematical exercises. It is true, as Ken Boston noted, mathematics has a higher 'technical demand' than natural language. In this it is more like music or modern foreign languages than English. However in those areas, too, real worthwhile performance is a curriculum goal – pieces of good music of various kinds, and practical conversations respectively. Mathematics is currently unique in its inward-looking focus on technique alone. This excludes not only functionality but all real mathematics.

**Dimensions of Balance in assessing Mathematics**

Returning to performance in mathematics, Table 1 overleaf sketches the *Dimensions of Balance* used in our task analysis. The model is, of course, far from unique. However, all the dimensions shown, not just the content one, are important in performance, and in its assessment. (We invite those who doubt this to identify any dimension that they think is not relevant)

[10] If it is assumed that adults will develop these attributes "in life and work", without being taught them in school, that is, to put it mildly, unwise; we know that most don't. They cannot be expected to. We do not normally try to justify neglect of teaching in any area of this importance.

## Dimensions of Balance

*Mathematical Content Dimension*

- **Mathematical content** in each task will include some of:

  **Number and Operations** including: number concepts, representations relationships and number systems; operations; computation and estimation.

  **Algebra** including: patterns and generalization, relations and functions; functional relationships (including ratio and proportion); verbal, graphical tabular representation; symbolic representation; modeling and change.

  Measurement including: measurable attributes and units; techniques tolls and formulas.

  **Data Analysis and Probability** including: formulating questions, collecting, organizing, representing and displaying relevant data; statistical methods; inference and prediction; probability concepts and models.

  **Geometry** including: shape, properties of shapes, relationships; spatial representation, location and movement; transformation and symmetry; visualization, spatial reasoning and modeling to solve problems.

*Mathematical Process Dimension*

- **Phases** of problem solving include some or all of:

  Modeling and Formulating;
  Transforming and Manipulating;
  Inferring and Drawing Conclusions;
  Checking and Evaluating;
  Reporting.

- **Processes** of problem solving, reasoning and proof, representation, connections and communication, together with the above phases will all be sampled.

*Task Type Dimensions*

- **Task Type** will be one of: design; plan; evaluation and recommendation; review and critique; non-routine problem; open investigation; re-presentation of information; practical estimation; definition of concept; technical exercise.
- **Non-routineness** in: context; mathematical aspects or results; mathematical connections.
- **Openness** –tasks may be: closed; open middle; open end with open questions.
- **Type of Goal** is one of: pure mathematics; illustrative application of the mathematics; applied power over a practical situation.
- **Reasoning Length** is the expected time for the longest section of the task.

*Circumstances of Performance Dimensions*

- **Task Length**: in these tests most tasks are in the range 5 to 15 minutes, supplemented with some short routine exercise items.
- **Modes of Presentation, Working and Response**: these tests will be written.

**Table 1.**

What do we learn from such a multi-dimensional analysis? For functional mathematics, it is absolutely critical that many of the assessment tasks are *non-routine*, asking students to find a solution path, not simply to remember one they have been taught for a closely similar problem – ie to think mathematically. This inevitably involves longer *reasoning length*. These, and other dimensions discussed below, are not found in current tests.

Papers A and B in the appendix exemplify (within our self-imposed rough limit of less than one hour per task) some of the types of task that functional mathematics should embrace, in both curriculum and assessment.

**Development methodologies and their limitations**

Designing balanced assessment is challenging. It requires a much higher standard of 'engineering', both in design skill and in systematic development through feedback from trialling, than for the imitative routine tasks that are currently assessed. The methodology of UK exam boards, mainly using part-time task designers working without evidence from trialling, does not work at this level[11]. The development methodology of Key Stage tests could be adequate, as KS3 ICT and to some extent English, have shown. The problems with Mathematics have (as we noted) come from the fragmentary model of performance they were asked to assess, in which it was assumed that assessing the parts assessed the whole. (In English this would have meant a test of spelling and syntax only)

The above comments are made on the basis of the analysis sketched above and, equally important, of our substantial experience over two decades in designing balanced assessment for UK exam providers, for QCA, and internationally.

> **Recommended actions:** Describe, in the top-level specification, the full range of performance targets in terms of a multidimensional analysis, exemplifying them via a balanced sample of assessment tasks with mark schemes – predominantly substantial functional mathematics tasks plus some relevant and credible subtasks.
>
> **Review** examples of assessment tasks, particularly from the 1980s, in deciding on the balance of task types that QCA will recommend.
>
> **Commission** in the light of these decisions, the development of high-quality tasks for assessment providers.

It is, of course, clear that these matters involve DfES as well as QCA. Particularly if the latter moves further into a purely regulatory role, the question of how and by whom active development will be handled will need clarification.

## Functional mathematics inevitably involves non-routine problems

At one time, reliable skills in, say, the arithmetic of money and the geometry of surveying equipped you for a lifetime of gainful employment. Everyone, and every employer, knows that these skills can now be bought as IT devices for around £100. The £10,000 cost of a school education in mathematics is no longer a good investment if it only produces reliable *automata* with these skills. Indeed human automata are losing their jobs all over the world. Now life regularly presents new problems; there is no way that people can learn to tackle all the kinds of problem they will face later – so we need to educate people who can *think with mathematics,* who have *functional mathematics.* In this section we look deeper into what this involves.

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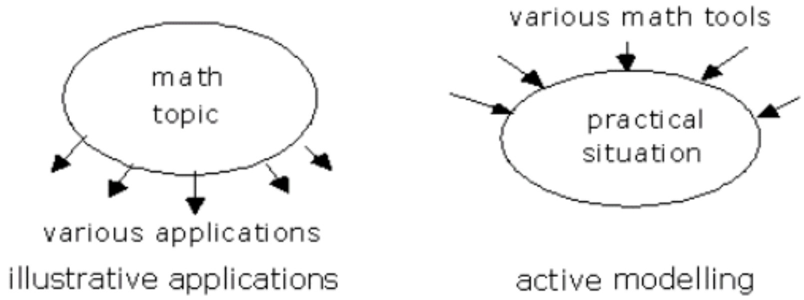
[11] We discussed with SCAA some years ago the advantages of QCA commissioning well-engineered 'common questions' to be used in all mathematics examinations at the same level. They would help in equating standards between different examinations, and in providing an *engine for improvement* to sustain and improve their range and balance. It would also reduce the need for QCA to micromanage the standard-setting. It seems is worth further consideration.

**Illustrative applications v active modeling**
Most curricula include some "applications", "using and applying" mathematics in real world contexts. However there is an important distinction between two different kinds of application, shown in the diagram.




Most curricula offer *illustrative applications*; there the focus is on a specific mathematical topic, showing the various practical domains where it can be useful and practising its use in those contexts. The student has no doubt as to the mathematics to be used – it is the topic just taught. In contrast, in *active modelling* the focus is on the practical situation and understanding it better. Usually, a variety of mathematical tools will be useful for different aspects of the analysis. (This is a good indication as to the real goal.) Choosing and using tools appropriately is a major part of the challenge to the student.

Both types of activity are important in learning functional mathematics. Both provide connections between mathematics and practical situations. However only active modelling, as opposed to learning standard models, involves the full range of competencies needed for functional mathematics – real life problems will often appear in a form that you have not be shown how to solve. In summary, FM is all about increasing *applied power* over the real world; at its centre is modelling with mathematics.

**What is modelling?**
We really should not have to answer this question, but there is confusion. Some see modelling is an abstruse high-level activity for professional applied mathematicians. Many teachers and others working in mathematics education see it as an obscure technical term of little professional concern to them. To some pure mathematicians, it is a basically trivial and uninteresting aspect of certain type of problem. These different view points reflect people's backgrounds and experience and, in particular, the neglect of modelling in most current school curricula..

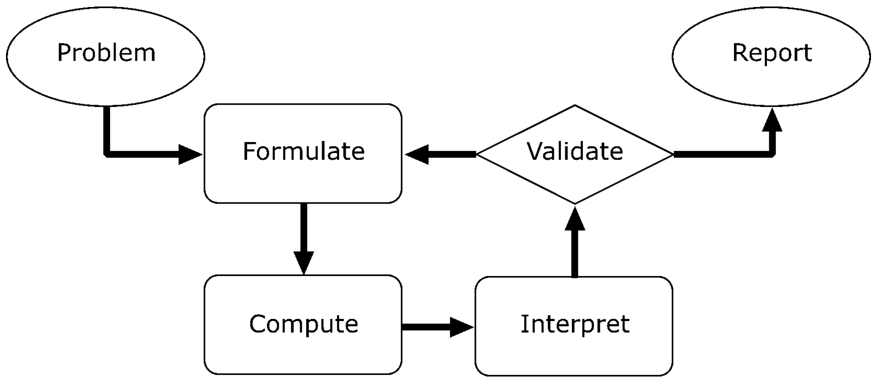
Modelling *is* still a technical term, rather like prose; people have been doing it all their lives but have not recognised it *or paid explicit attention to it*. Like prose, it exists at every level from the simple to the sophisticated – and is useful at every level.

Exemplification at a simple level will make the point most clearly. Consider the following three 'word problems':

a) Joe buys a six-pack of cola for £3 to share among his friends. How much should he charge for each bottle?
b) If it takes 40 minutes to bake 5 potatoes in the oven, how long will it take to bake one potato
c) If King Henry 8th had 6 wives, how many wives had King Henry 4th?

In current curricula, all the problems a typical chapter on proportional reasoning will be like a); the student does not have to choose an appropriate mathematical model – (s)he *knows* it is proportional, $y = kx$ or its numerical image. The only demand is to decide how to 'arrange the numbers', 3 and 6 in this case, among $y$, $k$ and $x$. b) requires active modelling – the appropriate model, and the answer, here depends on whether it is a conventional or a microwave oven. c) requires a recognition that there is no logical route to the answer, which is simply an independent fact. (Interestingly, c) comes from an early SMP book which included some modelling) The ability to choose an appropriate model is clearly crucial to functionality.

For mathematics to be functional it must include a substantial amount of modelling. What does this involve? This is not the place for a detailed description of the modelling process; we shall simply illustrate it with the standard top-level diagram of the *phases*.




**The phases of modelling**

Each of these phases is an integral part of functional mathematics. In the above trio of examples, we have discussed the *formulation* phase, the choice of an appropriate mathematical representation for the situation. This is often the most challenging; the student has to *think about the real world with mathematics*, rather than just remember a procedure (s)he has been taught.

**Current v functional mathematics curricula**

Most current curricula focus overwhelmingly on the *solve* phase – *formulation* is absent or, as above, very closely guided. *Interpretation* and *evaluation* of the results is largely absent, except in some *Data* tasks[12]. *Reporting*, so important to employers, is only ever asked for in coursework. (Explanation marks in test questions are used to give partial credit to students who have the *wrong* answer; there is no credit for the ability to explain your results and reasoning clearly – a crucial aspect of FM)

It may appear from this analysis that learning and teaching real problem solving is a formidable challenge. However, research and development over the last forty years has shown how to enable typical teachers to teach it effectively. How this is done, and the demands on teachers, are outlined below[13].

Modelling turns out to have important spin-offs for learning mathematics itself (see e.g. Blum et al, 2006, section 3.4). These include improved understanding, reliability and

[12] Statistics education often includes more of the modeling phases, and links to the real world. Their wish to remain separate from mathematics education is thus understandable. However, functional mathematics requires them to be integrated – sometimes a deterministic model, conceptually simpler, will do but often random variation must be taken into account, so statistics is essential.

[13] They do have to know at least as much as their students about the world outside the mathematics classroom, and use it in their teaching. Some maths teachers feel they should only teach mathematics – but can one imagine an English teacher, say, regarding the real world as other than an asset for their teaching?

motivation. Many more students, when they see that mathematics gives them increased power over other aspects of their lives, become interested in learning more – a key concern, highlighted in the Smith Report. A number of studies have shown that 'solving word problems' by *direct translation* of the words into mathematics is unreliable, essentially because of the flexibility of natural language; only a *comprehension-modelling* approach, in which the problem situation is understood and then modelled with mathematics, is generally effective. This explains the difficulty, to some surprising, of word problems – further, their concise language can make comprehension more difficult.

**Assessing non-routine problems**

Some are concerned that it is impossible, or unfair, to assess non-routine problem solving, particularly in timed examinations. There is much practical experience that shows these concerns are unfounded.

It is true that teachers feel more secure when they know that every assessment task will be familiar to their students. Unfortunately this encourages a teaching approach focused on learning routines, not on functional performance.

Non-routine problems do present some interesting design challenges, for both teaching and assessment materials. Here we will mention just two.

- In assessment it is useful to control the *transfer distance*, the degree of unfamiliarity of the task, which is an important factor in difficulty. There are various methods for doing this.
- What kinds of situation should students be asked to model? Situations that students know well informally but have never analysed are particularly good. Computer simulations where the student can easily explore the behaviour of a system can also work well. Problems from other school subjects are less often suitable – many students simply don't understand enough about the situation to model it.

**Recommended action:** Make clear, in the top-level specification, that functional mathematics involves learning to tackle non-routine problems from the real world – and that this will be assessed. Examples will be crucial in communicating what you mean.

## Progression between levels is multi-dimensional

**Progression between levels** in the current draft is given entirely in terms of content, the list of mathematical topics to be covered. It does not reflect the principle:

> "…they will have the confidence and capability to use maths to solve problems embedded *in increasingly complex settings* and to use a range of tools, including ICT as appropriate."

That is the way that functionality progresses.

There are at least three dimensions to difficulty, and progression:

- the complexity of the practical situation being analysed
- its unfamiliarity to the solver
- the technical tools used in tackling it (ie the content)

The level of challenge of a task depends on a complex combination of these. It cannot be reliably predicted but can be determined empirically through trialling. Stressing content alone is easy but will preclude developing functionality.

**"The few-year gap"**

It has been found in work on introducing non-routine problem solving that the mathematical concepts and skills that students can use autonomously in tackling non-routine problems (pure or practical) is typically a few years behind those they are learning and practising in imitative exercises[14].

This is not surprising, since identifying a mathematical tool as useful for a problem depends on seeing multiple connections for that tool, while using it effectively and reliably depends on a high level of mastery. Both of these only develop over time.

They develop more quickly with problem solving experience that builds such connections. Conventional teaching, in contrast, tends to link a new topic to the one before – on its own an inadequate basis for the robust understanding that functionality requires.

**Narrowing the range of performance**

There is good news. We have just noted that most students who perform at a high level in the current imitative curriculum use *less* mathematics in tackling non-routine problems. However, for many low-performing students, we and others found that the reverse is true (Shell Centre 1987-89) – they show far more mathematical power in functional mathematics. This seems to be because they regard school mathematics as irrelevant to them and their lives; this attitude shifts dramatically when they tackle *real problems*, *credible as such to them*, with mathematics.

Thus, from a functional perspective, the test results of high-achievers may be regarded as 'false positives', and for the low-achievers as 'false negatives', in showing what they really *know, understand and can do* with mathematics. Given the current exceptionally wide range of performance when compared with other subjects that Ken Boston noted, this should be regarded as good news.

On this basis, we believe the skills that are in the current draft specifications, especially at E1 to E3, are insufficiently ambitious. They reflect imitative curricula, for students at school and adults, that teach basic skills rather than real problem solving with mathematics. This question can be resolved by practical comparative studies using existing curriculum elements of both types.

> **Recommended actions:** Make the multi-dimensional nature of progression clear in the descriptions on page one of each Level, mainly by substantial exemplification of the target kinds of performance tasks for that Level. (see above)
>
> The content descriptions for each level should then, along with earlier content, be the *maximum* content that may be used in real problems at that level. Content, particularly that not tested in the real problems used, can be tested in short items. (The 'few year gap' should be taken into account in defining the content – it may be in the current list, at least for lower Levels)

## The implications for teaching are profound

Understandably, the implications of the proposals for the implemented curriculum in classrooms are not discussed in the current drafts. However, these are crucial to successful implementation and need to be thought through at this stage. The changes in the balance of classroom activities needed to achieve functionality (or, indeed, any not-purely-imitative performance) are substantial. They are briefly summarised in the Cockcroft Report, paragraph 243, and in many other research-aware reviews of mathematical education. They are described in the non-statutory documents of the National Curriculum. However, for the reasons we have discussed, they do not now appear to feature in many mathematics classrooms.

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[14] One example: In a study of real problem solving by 120 able 17-year old students who were all expected to get two A's in double maths at A-level, Treilibs et al (1980) found that, while some were effective at modelling with arithmetic and graphs, not one used algebra – in which all had 5 years of success in the imitative curriculum. These students had not been taught modelling; there is some evidence from work at undergraduate level that the few-year gap can be eliminated by intensive teaching of modelling with mathematics.

We think it worth summarising, particularly for those not familiar with the research in cognitive science and mathematical education of the last 30 years, the key elements of the changes that are essential if functional mathematics is to be delivered.

**Essential skills beyond the 'basic'**

Modelling involves all the key aspects of 'doing mathematics' which may be summarised (see e.g. Schoenfeld 1992) as:

**knowledge** of concepts and skills

**strategies** and tactics for modelling with this knowledge

**metacognitive control** of one's problem solving processes

**disposition** to *think mathematically* about practical problems, based on **beliefs** about maths as a powerful 'toolkit' (rather than just a body of knowledge to be learnt).

These are not, of course, independent elements but must be integrated into coherent **modelling practices** for tackling whatever problem is at hand.

In the context of functionality of mathematics, this has a specific implication – *the explicit teaching of modelling with mathematics* discussed above. (This is implied in the current draft standards; again, much more explicit exemplification, and support, will be needed to give meaning to the principles). What does this require? How do we get curricula that develop all these elements? What support will teachers need? We address these key questions in turn.

**Richer learning activities**

To develop this range of skills, the main classroom elements we need, beyond those found in traditional curricula, are:

**active modelling** with mathematics of non-routine practical situations;

**diverse types of task**, in class and for assessment;

**students taking responsibility** for their own reasoning, and its correctness;

**classroom discussion** in depth of alternative approaches and results;

and, of course, **teachers with the skills** needed to handle these activities.

These imply a profound change in the *classroom contract*, the set of mutual expectations between teacher and students as to their respective roles and actions (see Brousseau, 1997). Table 2 (Burkhardt et al, 1988) illustrates the necessary role changes:

**Table 2. Teacher and Student Roles**

| for imitative learning | for modelling, add |
|---|---|
| **Directive roles** | **Facilitative roles** |
| Manager | Counsellor |
| Explainer | Fellow student |
| Task setter | Resource |
| | |
| *with students as* | *with students as* |
| *Imitator* | *Investigator* |
| *Responder* | *Manager* |
| | *Explainer* |

**Broader teaching strategies**
What extra skills do teachers need to make this a reality? The key elements here include:

- **handling discussion** in the class in a *non-directive* but supportive way (see e.g. Swan et al. 1986, inside back cover), so that students feel responsible for deciding on what to try, and the correctness of their and others' reasoning – rather than expecting either answers or confirmation to come from the teacher;

- **giving students time and confidence** to explore each problem thoroughly, offering help only when the student has tried, and exhausted, various approaches (rather than intervening at the first signs of difficulty);

- **providing strategic guidance** and support, without structuring the problem for the student or giving detailed suggestions on what to do (see e.g Shell Centre 1984, inside back cover);

- **finding supplementary questions** that build on each student's progress and lead them to go further.

This is challenging for teachers at first, but those who acquire these skills continue to use them across their teaching. Well-engineered materials can provide enormous support to teachers and students who are engaging in learning modelling. Indeed, such materials are essential for most teachers in their first few years of such teaching, if they are to succeed. They can be further supported by ongoing professional development that is closely linked to teaching strategies and classroom examples.

The National Numeracy Strategy is a recent example of the kind and scale of effort that will be needed for these more ambitious goals. Though we believe that its design and development methodology could be improved on[15], it provides a model for planning.

> **Recommended actions:** Make it clear to the Department for Education and Skills that, if functional for most people, the development of a substantial range of *support* for teachers and assessment provides will be needed – that *pressure* alone will not be enough.

## Realising functional mathematics is a major challenge

One gets the impression that policy makers believe that deciding on curriculum changes is difficult but, once made, that implementation of the decisions is straightforward[16]. This is far from reality for substantial changes of the kind we are discussing here. Gross distortion of the intentions is common, largely because the challenge of implementation is underestimated. The change model needs to be as well-engineered, i.e. as carefully designed and developed, as the assessment, curriculum and other elements within it. In this, it is worth looking at models that have proved fairly successful. The following suggestions are made on that basis.

**Pace of change**
This is a central problem that is rarely even discussed. Here we shall sketch the main issues and possible ways forward. In this context the policy question is:

> *Over how many years should functional mathematics assume its full intended role in mathematics curricula.*

The policy temptation is to look for "immediate" implementation, typically over a year or two. "If this is what students need, we should not delay." Yet the timescale of

---

[15] Would you let your children fly in an airplane, or take a new medicine, that had been developed in the same way as most educational innovations. (See Burkhardt 2006a)

[16] In a break at a meeting of the National Curriculum Working Group, I asked a senior civil servant whether she thought the decisions would be implemented in schools. "Of course", she said, "It is the Law of the Land".

educational change is a decade or two – as exemplified, for example, by the development of the National Curriculum since 1987. How can this be reconciled with the few-year timescale of politics, driven as it is by elections and media impatience for results (preferably failure!)?

**Gradual or Big-Bang change?**

While immediate implementation of a preferred solution has clear attractions, for outcomes that are close to intentions there is much to be said for gradual step-by-step change – a medium term (5 year, say) goal and short-term year-by-year targets moving towards it. It has advantages:

for politicians, it generates annual initiatives that, provided the targets are credible to the public, they can proudly present, and soon see success[17].

for teachers, the demands are digestible. (We have found that about three weeks of challenging new teaching with a class, provided it is supported by clear performance targets and well-engineered teaching materials, are enjoyable – as is the return to more familiar territory for a while at the end.)

for assessment providers, the introduction of one new task-type each year provides continuity of standards from year to year.

for the education system as a whole, it provides a realisable model of change, *which can learn as it proceeds* without a wholesale revamp. (This last point, design to learn from feedback, is crucial for the success of any complex adaptive system)

This is an area worth consideration, including a review of past initiatives – and some experiment.

**Teaching and assessment materials**

It has been repeatedly shown that materials can enable typical teachers to realise ambitious teaching and learning goals, *provided they are well-engineered*. Good engineering is:

- imaginative design on the basis of prior research;
- careful development through successive rounds of trialling in increasingly realistic circumstances, learning from detailed feedback on the trials;
- subsequent evaluation in depth of use in the field, guiding the improvement of subsequent versions and variants.

This '*engineering research*' approach is standard in other applied fields. It is more expensive, and more demanding of skill[18], than the *craft-based* methods that are common in education, usually based on simply sharing the authors' successful experience. The investment will be repaid many times over in reduced costs of successful implementation.

The same is true of assessment materials which, for reasons discussed above, need to assess, with acceptable reliability, *all the core elements of functional mathematics*, as outlined above and illustrated in the Appendix.

**Professional development support**

Here we tread on more controversial ground. It is common to rely on professional development as the prime method of support for implementing new initiatives. Though it undoubtedly has a useful role to play, it does not seem to be the most cost-effective approach. Live, ongoing professional development on the scale shown to make a difference is expensive in teacher time (the largest cost in any education system). It

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[17] A change that introduces new, valuable performance targets like functional mathematics (rather than tackling old ones, like addition of fractions, in a different way) almost guarantees success over the first few years.

[18] There is a need in this country to build capacity in this area. In the Netherlands, for example, the Freudenthal Institute has 40 fte researchers doing such work in a coherent programme with Government support over many years; Dutch performance in the international tests reflects this.

also relies on expert leaders, who are in short supply. Further, what evidence there is suggests that much professional development has limited success in *changing teachers' classroom behaviour* – the core of what is needed to teach functional mathematics effectively. This is partly because PD is usually evaluated simply by asking teachers how satisfied they feel about the experience, on which the score is often high.

When observational feedback on classroom behaviour[19] is part of the development, a very different kind of PD often emerges – with more emphasis on specific classroom activities, built around giving teachers a sequence of successful experiences in their own classrooms, with opportunity to reflect on them, and discuss them with others. Principles emerge from these discussions – essentially, a constructivist approach to *teacher* learning.

> **Recommended actions:** A gradual approach to implementation should be adopted, at the fastest pace that enables teachers in typical classrooms really to equip students with functional mathematics.
>
> Existing materials should be evaluated in depth in classrooms and through tests. Materials should be developed using engineering research methods so as to work well with typical teachers of Mathematics, and in the planned range of assessment conditions. Implementation should follow this development phase.
>
> Professional development activities should be well-engineered and *shown* to enable typical teachers to achieve the required changes in classroom behaviour. It should be supported by materials that enable less-experienced PD leaders to achieve this.

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[19] Some leaders of the profession regard such 'intrusions' as an infringement of teachers' rights as independent professionals. This view of professional development, as "a civilised exchange of views between fellow professionals", ignores the wide range of performance in teaching, characteristic of any skilled activity but uncomfortable for the standard professional posture – that all qualified practitioners are 'skilled'. Equally for the public, no-one likes to think their doctor, their lawyer, or their teacher is mediocre, though many are. Government should tackle reality in this, as in other, areas.

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# Functional Mathematics
# at 'AS-Level'

A few thought-provoking tasks
that any well-educated adult could, and should, be able to do at AS-Level
*(without having been taught the specific problem).*

Currently, many can't
*(see commentary on page 4).*

*For a 'serious' test, these tasks would need further trialling and refinement*
*probably Levels 2 and 3 for well-taught students*

from
MARS: Mathematics Assessment Resource Service
Shell Centre for Mathematical Education
University of Nottingham

Summer 2005

**Sudden Infant Deaths = Murder?**

In the general population, about 1 baby in 8,000 dies in an unexplained "cot death". The cause or causes are at present unknown. Three babies in one family have died. The mother is on trial. An expert witness says:

*"One cot death is a family tragedy; two is deeply suspicious; three is murder. The odds of even two deaths in one family are 64 million to 1"*

Discuss the reasoning behind the expert witness' statement, noting any errors, and write an improved version to present to the jury.

**Conference budget**

Your job is to plan a conference budget, using a computer spreadsheet.

You have already made a start:

| A | B | C | D | E |
|---|---|---|---|---|
| | **College charges** | **Number** | **@ £ each** | **£** |
| **Monday** | Buffet Supper | 30 | 17.00 | **0** |
| | Single En-suite Accommodation | 30 | 40.00 | **0** |
| **Tuesday** | Breakfast | 30 | 8.00 | **0** |
| | Morning Coffee | 30 | 1.90 | **0** |
| | Luncheon | 30 | 15.00 | **0** |
| | Afternoon tea | 30 | 1.90 | **0** |
| | Dinner served | 30 | 50.00 | **0** |
| | Single En-suite Accommodation | 30 | 40.00 | **0** |
| | Plenary Room | 30 | 15.77 | **0** |
| | Breakout rooms | 2 | 85.10 | **0** |
| **Wednesday** | Breakfast | 30 | | **0** |
| | Morning Coffee | 30 | | **0** |
| | Luncheon | 30 | | **0** |
| | Afternoon tea | 30 | | **0** |
| | No Dinner | 30 | | **0** |
| | Single En-suite Accommodation | 30 | | **0** |
| | Plenary Room | 30 | | **0** |
| | Breakout rooms | 2 | | **0** |
| **Thursday** | Breakfast | 30 | 8.00 | **0** |
| | Morning Coffee | 30 | 1.90 | **0** |
| | Luncheon | 30 | 15.00 | **0** |
| | Afternoon tea | 30 | 1.90 | **0** |
| | Dinner | 30 | 17.00 | **0** |
| | Single En-suite Accommodation | 30 | 40.00 | **0** |
| | Plenary Room | 30 | 15.77 | **0** |
| | Breakout rooms | 2 | 85.10 | **0** |
| **Friday** | Breakfast | 30 | 8.00 | **0** |
| | | | Total charges | **0** |
| | | | VAT | **0** |
| | | | **Total** | **0** |

(i) Complete the entries for Wednesday in column D.

(ii) Calculate appropriate totals in column E.

*(The spreadsheet was on a computer; here, work out what you would do)*

**Primary teachers**

In a country with 60 million people, about how many primary school teachers will be needed? Try to estimate a sensible answer using your own everyday knowledge about the world. Write an explanation of your answer, stating any assumptions you make.

**Bike or Bus**

Terry is soon to go to secondary school. The bus trip to school costs 50p and Terry's parents are considering the alternative of buying a bicycle.

Help Terry's parents decide what to do by carefully working out the relative merits of the two alternatives.

**Scheduling Traffic Lights**

A new set of traffic lights has been installed at an intersection formed by the crossing of two roads. Right turns are NOT permitted at this intersection.

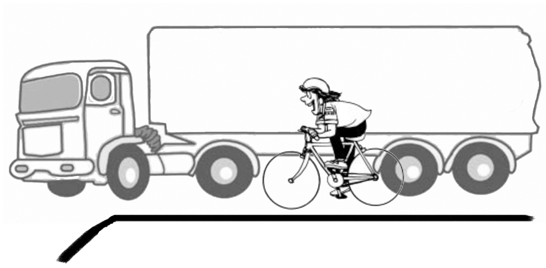
For how long should each road be shown the green light?

Explain your reasoning clearly.

**Left turns**

The lorry is stopped at traffic lights, planning to turn left. The cycle is alongside.

If the cyclist waits for the lorry to turn before moving, what will happen? Explain why this will happen with a diagram.



What would be your advice:

- to the lorry driver?
- to the cyclist?

Give reasons in each case.

**Being realistic about risk**

"My sixty year old mother, who lives in London, gets frightened by newspapers. One day she is afraid of being a victim of crime, the next she is frightened of being killed in a road accident, then it's terrorists, and so on."

(i) Use the Office of National Statistics website to estimate the chances of my mother being a victim of the above events, and others you think she might worry about.

(ii) Write down some reassurance you would give her – and compare the likelihood of these events with the probability that women of her age will die during the coming year.

## *Commentary on the tasks, and responses to them*

**Sudden Infant Deaths = Murder?**
What we expect here is not a full statistical analysis, which would need more information, but a recognition that the reasoning presented is deeply flawed. There are two elementary mistakes in the statement, and one that is a bit more subtle. It would be correct to say:

1. The chance of these deaths being entirely *unconnected* chance events is very small indeed – if there has been one death, the chance of two more unconnected deaths is about 64 million to one.
2. What can the connection be? It may be that the mother killed the children; on the other hand, particularly since we do not understand the cause(s) of cot death, there may be other explanations. For many conditions (cancer and heart disease, for example) genetic and environmental factors are known to affect the probability substantially.

Any lawyer or judge with functional mathematics should have seen problems with the witness statement. It is not lack of basic skills that was their failing (They could surely have worked out the chance of a double six on rolling two dice as 1/36) but an understanding of the necessary assumptions.

**Conference budget**
This is a task we give (on a working spreadsheet) to candidates for the post of Secretary/Administrator in the Shell Centre team. Most are graduates. All "know Excel". None complete the task. Most see that Wednesday's values in Column D are probably the same as Tuesday's and Thursday's. Few enter the appropriate formulas, or indeed any, in Column E (Formulating relationships is a basic piece of algebra that is neglected in schools – and maths tests). Some even work out the row totals on a calculator, row by row, entering the *values*!

**Primary teachers**
This kind of back-of -the-envelope calculation is an important life skill. Here it requires choosing appropriate facts (6 years in primary out of a life of 60-80 years, one teacher for 20-30 kids), and recognizing and using a proportional relationship giving (60*6)/(70*25) = 0.2 million primary teachers (to an accuracy appropriate to that of the data) This kind of linkage with the real world, common in the English curriculum, is rare in school Mathematics (and absent in tests)

*Bike or Bus* and *Scheduling Traffic Lights* – as for Ice Cream Van on the other test.

**Left turns** Functional mathematics often involves space and shape, too.

**Being realistic about risk**

Education, and functional mathematics in particular, can help narrow the gap between perceived and real risk. Given the power of anecdote over evidence, exploited daily by the media, this is a major challenge; meeting it could make a huge contribution to people's quality of life, and that of their children. As well as no sense of the magnitude of specific risks, few people have any idea of the 'base risk' for someone of their age. (Note that only order-of-magnitude estimates, not accurate numbers, are relevant here)

Explicitly teaching students to use their mathematics on real problems is now proven, with typical teachers; it is essential to functionality. These exemplars also show how deterministic and statistical reasoning intermesh in functional mathematics.

# *Functional Mathematics at 'GCSE'*

A few thought-provoking tasks
that any well-educated adult could, and should, be able to do at GCSE
*(without having been taught the specific problem).*

Currently, many can't
*(see commentary on page 4).*

*For a 'serious' test, these tasks would need further trialling and refinement*
*probably Levels 1 and 2 for well-taught students*

from
MARS: Mathematics Assessment Resource Service
Shell Centre for Mathematical Education
University of Nottingham

Summer 2005

**At the airport**

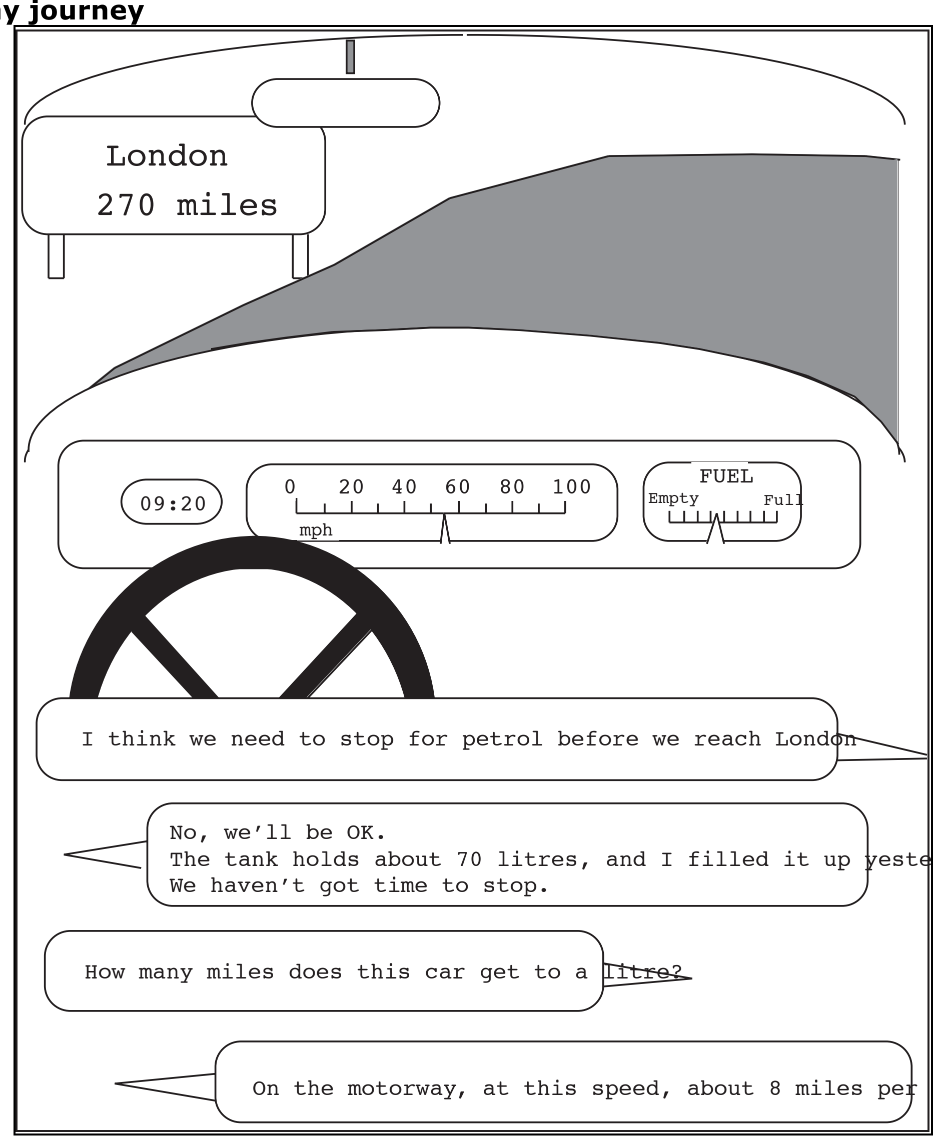
| Den's Currency Exchange | | |
|---|---|---|
| Currency | We Buy | We Sell |
| **$** US Dollar | £ 0.533 | £ 0.590 |
| **€** Euro | £ 0.660 | £ 0.730 |
| No commission! | | |

(a) How many Euros (€) would you get for £500?

(b) How many Pounds (£) can you get for $700?

(c) How much would you have to pay, in Pounds and Pence, to get exactly €550?

**Motorway journey**




(i) Do they have to stop for petrol? Explain your reasoning.

(ii) Suppose they decide to stop for 10 minutes.
At what time will they reach London?

**Ice cream van**

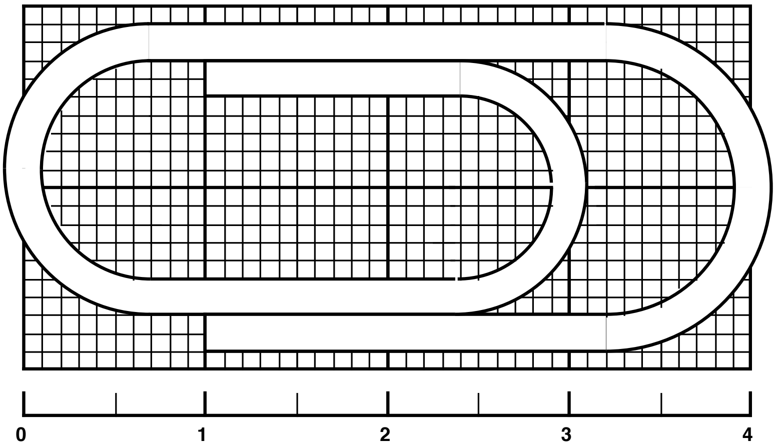
You are considering driving an ice cream van during the Summer break. Your friend, who "knows everything", says that "It's easy money". You make a few enquiries and find that the van costs £600 per week to hire. Typical selling data is that one can sell an average of 30 ice creams per hour, each costing 50p to make and each selling for £1.50.

How hard will you have to work in order to make this "easy money"?

Explain your reasoning clearly.

**Paper clips**

This paper clip is just over 4 cm long.




How many paper clips like this may be made from a straight piece of wire 10 metres long?

**Cold calling**

*The following is part of a genuine letter of complaint to a bank.*

*"I would like to complain about the behaviour of XYZ Bank and the advice given during a recent unsolicited telephone call. Having been told I was "pre-approved" for a £5,000 loan, the operator asked me for my financial details. I told her that I currently had two credit cards, one with a balance of £3000 and one with £1000. She said that they could consolidate these debts into a single payment which would be cheaper. I pressed her on the APR which she explained was 16.4%, which caused me to decline the loan because my two credit cards are currently at 7% and 9.9% APR respectively. The operator then informed me that their loan would work out cheaper, because 7% and 9.9% works out at 16.9%, nearly 0.5% higher than the bank loan."*

(i) Explain what is wrong with the operator's reasoning.

(ii) How much more expensive is the bank's consolidated loan?

**Commentary on the tasks:**

**At the airport**
It is interesting to compare this with a question from a current GCSE paper:

The table shows the exchange rates between different currencies:

| | | |
|---|---|---|
| £1 (Pound) | is worth | € 1.45 (Euros) |
| $1 (Dollar) | is worth | € 0.81 (Euros) |

(a) Jane changes £400 into euros. How many euros does she receive?

(b) Sonia changes £672 euros into dollars. How many dollars does she receive?

Note how the simplification of the presentation leaves a major gap from real functionality. This unreality, characteristic of secondary school mathematics, confirms many students' view that the subject has no relevance to their lives.

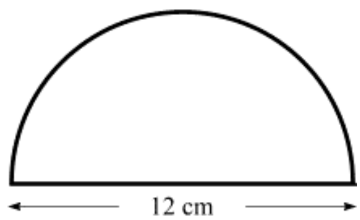
**Motorway journey**
From an actual test. Most examples of functional mathematics have been eliminated in the fragmentation of tasks to assess separate micro-skills.

*Ice cream van*

This task was used in a research study of the performance of 120 very able 17 year old students. Many solved the tasks, using arithmetic and, sometimes graphs. *None used algebra*, the natural language for formulating such problems. Their algebra was non-functional, despite 5+ years of high success in the standard imitative inward-looking algebra curriculum.

**Paper clips** – *exemplifies a step towards functionality; a GCSE version is:*

(b) A semi-circle has a diameter of 12 cm. Calculate the perimeter.




**Cold calling –** *a common misconception, and con, to unravel.*

Explicitly teaching students to use their mathematics on real problems is now proven, with typical teachers; it is essential to functionality. These exemplars also show how deterministic and statistical reasoning intermesh in functional mathematics.