Modelling in Mathematics Classrooms: reflections on past developments and the future

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Abstract: *This paper describes the development of mathematical modelling as an element in school mathematics curricula and assessments. After an account of what has been achieved over the last forty years, illustrated by the experiences of two mathematician-modellers who were involved, I discuss the implications for the future – for what remains to be done to enable modelling to make its essential contribution to the "functional mathematics", the mathematical literacy, of future citizens and professionals. What changes in curriculum are likely to be needed? What do we know about achieving these changes, and what more do we need to know? What resources will be needed? How far have they already been developed? How can mathematics teachers be enabled to handle this challenge which, scandalously, is new to most of them? These are the overall questions addressed.*

The lessons from past experience on the challenges of large-scale of implementation of profound changes, such as teaching modelling in school mathematics, are discussed. Though there are major obstacles still to overcome, the situation is encouraging.

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The last 40 years have seen the explicit teaching of modelling with mathematics move forward, from small-scale explorations, through developments in typical classrooms, to established courses, albeit in a small minority of mathematics classrooms worldwide. Two kinds of student learning activity are essential for mathematics to be functional in everyday life and work:

learning illustrative applications and standard models;

active modelling by students, using mathematics to tackle problems that are new to them.

There are now some examples of well-engineered materials to support both activities. However, there are important gaps. We need to know more about how best to enable *all* teachers of mathematics to acquire the extra mathematical and pedagogical skills needed to teach modelling. Further, while policy makers regularly stress the importance of students "being able to use their mathematics", their actual policy decisions rarely support effective ways to bring this about. This paper will review the current situation and outline a programme for further progress.

1. Modelling with mathematics

For this issue of ZDM, I do not need to describe the processes of modelling in detail; it is common ground. However, discussion of the nature of modelling and its role, both in mathematics and in mathematics education, is often oversimplified. For mathematics education this arises partly because the dominant intellectual influences on school mathematics have come from pure mathematicians. Understandably, they regard it as 'their subject'; yet their attitudes to mathematics differ fundamentally from the far greater number of those who use mathematics in life and work, as citizens or as professional users of mathematics in engineering, science, economics and other fields where mathematics is a key language. These all see mathematics as primarily a powerful *toolkit* to help them understand and solve problems from the real world. That will be the perspective of this paper. While the 'pure' and 'applied' viewpoints have many things in common, including delight in the elegance of mathematics, they differ in others that are important for the design of school curricula – notably the central importance of teaching modelling.

To bring out the aims of modelling in mathematics education, I will begin with personal views on modelling as an activity from two mathematician-modellers who work in mathematics education, Henry Pollak[1] and myself. These comments reflect our different experience of 'doing mathematics', but illustrate the attitudes

[1] I am grateful to Henry, not only for his substantial contributions to this paper through 'private communications' but for illuminating discussions on the subject over many years.

to mathematics we seek to develop in students, and the adults they become.

HOP: My problem has always been that I like too many different areas of mathematics. As an undergraduate at Yale, I liked everything, but particularly analysis and topology of the old-fashioned point-set kind, and any combinatorics and number theory I saw, even though we had almost no courses in those kinds of areas. My enjoyment for topology came from Ed Begle. Of course, that isn't the way you get a doctorate – there you have to find a hole and dig and dig until it's yours and yours alone. I went to graduate school at Harvard, where I thought that I could do more topology with Hassler Whitney – but he soon left and I got into Complex Variables, and did a thesis in Geometric Function Theory with Lars Ahlfors. I also worked on a Navy Contract under Stefan Bergman in the Engineering School during most of my time there. In my last year of graduate work, I taught.

When it came time to find a job, it was clear to me that, even though everybody *said* that you were going to be paid to do good teaching, don't let anybody kid you – you were really going to be paid for your research. I thought that was kind of dishonest. A recruiter from Bell Laboratories came by and I went to see him. He made it clear that if I were going to be hired, it would be to do research, and that sounded good. I was probably the first mathematician hired there in a long time who was trained as a pure mathematician. But it worked out OK.

Alongside helping with various people's math problems, mostly in classical analysis, I started working in the area of balancing and optimizing a defensive missile system. I made a simplified mathematical model of it, which in retrospect I know now was a dynamic programming model. I didn't know about Richard Bellman's work (I don't even know if that came earlier); I therefore didn't know that nobody ever solved dynamic programming problems analytically – and so I did just that. Bell Labs appreciated it.

I also learnt that I wasn't supposed to write papers as tersely as possible, in math journal style; I was writing for people who could use the understanding and techniques I might have developed, *and how I got there*. So I learnt to take the time to document it in such a way that they could understand all of that – another link to teaching.

Bell Labs in its heyday was full of interesting people, with exciting things going on. Everybody was encouraged to talk to everybody else. The most interesting things I did might come from anywhere – from a specific practical question, or from some idea in the atmosphere that seemed not to be fully understood. Broad interests and good communication were at the heart of it all.

HB: My perspective on modelling comes both from my experience in using mathematics in tackling diverse problems from everyday life and from my university experience as a theoretical physicist – *we* are all modellers, particularly in forefront fields of physics like my own, elementary particles, where there is no established theory[2].

Even where there is an underlying theory that is true, the complexity of a system may make it insufficient. Turbulent flow of liquids and gases is a classical example where you cannot calculate directly the consequences of Newton's Laws; you have to build models. Quantum mechanics provides others. It works well for atoms, where there is a heavy central nucleus with a large dominant electric charge around which the light electrons move, essentially independently. Neither of these conditions applies inside nuclei; though quantum mechanics applies there too, there is no dominant centre and the particles interact strongly with each other. This situation makes the calculations too complex for current methods. Nuclear theorists developed a set of models, each of which accounts for important features of the data; however, their assumptions seem contradictory. The 'Shell Model' assumes that the particles move independently around a common force centre, as in an atom. How can this be so? Yet it explains the quantum states

[2] A theory is a model that has been shown to work well across a well-defined field of phenomena: Newton's Laws of Motion, or Gravitation, or Maxwell's Electromagnetic Theory, Darwin's Theory of Evolution, and Quantum Mechanics are giant examples; there are many more limited theories – of gas pressure, circulation of the blood, or elasticity, for example – effective within smaller well-defined domains of phenomena.

found in light nuclei. In contrast, the 'Liquid Drop Model' treats the nucleus as a liquid with every particle interacting strongly with its neighbours alone; this works well for some properties of heavy nuclei. The reconciliation of these and other apparently incompatible models is an interesting story – but it is not for here.

The point from our perspective is that limited precision and inconsistency are characteristic of modelling in all fields, profound or familiar, abstract or concrete. Economics is equally full of different perspectives that fit some features of the system but not others; the inconsistencies stimulate deeper analysis. People seek models that seem promising to explain features of the data, but they are often incomplete. When Watson and Crick were trying to work out the structure of DNA, Franklin's X-ray pictures suggested to them a helical structure. They then experimented with various wire models of the molecular structure (literal modelling, indeed) until they found one that would reproduce that X-ray data and fit the laws of chemical bonding. Their creation-discovery of the double helix model has been a beacon of scientific research, and medical advance, ever since but, though the essence of the structure was correct, there were complexities in the way it works that stimulated decades of further modelling by molecular biologists.

How does all this relate to everyday life problem solving of ordinary people – to the student wondering whether she should go to university or straight into employment, for example? The parallel is direct. There is no single formula that will provide the answer. There are various factors and perspectives. She may choose to compare estimates of lifetime earnings – graduates earn more but start later. She will have to balance short and long term issues – her immediate need for money against investment in the future. What will it cost to take time out to look after children, and what help will the state (or her family) provide? Over and above these important aspects are her quality of life considerations. Does she *want* to study? What does she think of various long-term plans she might consider? Can she keep the freedom to change her mind at each stage, and at what cost?

Many people see the complexity of a problem they face, and simply make a 'gut decision'; more effective thinkers do the analysis first – and then make a 'gut decision'. This means modelling. Even for complex problems like this, simple models can illuminate the analysis. Learning to model, as with all high-level skills, begins with relatively simple analysis and builds from there.

In summary, learning to model with mathematics is at the heart of learning to do the analysis that guides understanding and sensible decisions. What does it involve? Figure 1 shows an early version (Burkhardt 1964, 1981) of the now-familiar modelling diagram. It has some features that are not always shown – the *simplification* and *improvement* loops which reflect the discussion above, and the duality of the phases of the modelling process and the states of the problem.

Of course, like any other such diagram, this is a simplification of the modelling process. Particularly for skilled modellers, modelling actually involves 'look ahead' and 'look back' interactions between the different phases – they do not want to *formulate* models that they can't *solve* (though sometimes they find they have to). Treilibs et al (1980) give a more detailed analysis of the formulation phase that illustrates this – but it, too, is still a simplification. However, this does not mean that these 'models of modelling' are not useful. They provide insights and a structure that helps people to monitor their problem solving as it proceeds – or gets frustrated[3].

[3] Here, as so often, choosing the right 'grain size' for analysis is important – one can look in too *much* detail. In understanding gas pressure, for example, it does not pay to look at every molecule's motion. In planning a nation's economy, it does not help to look at each individual buying decision.

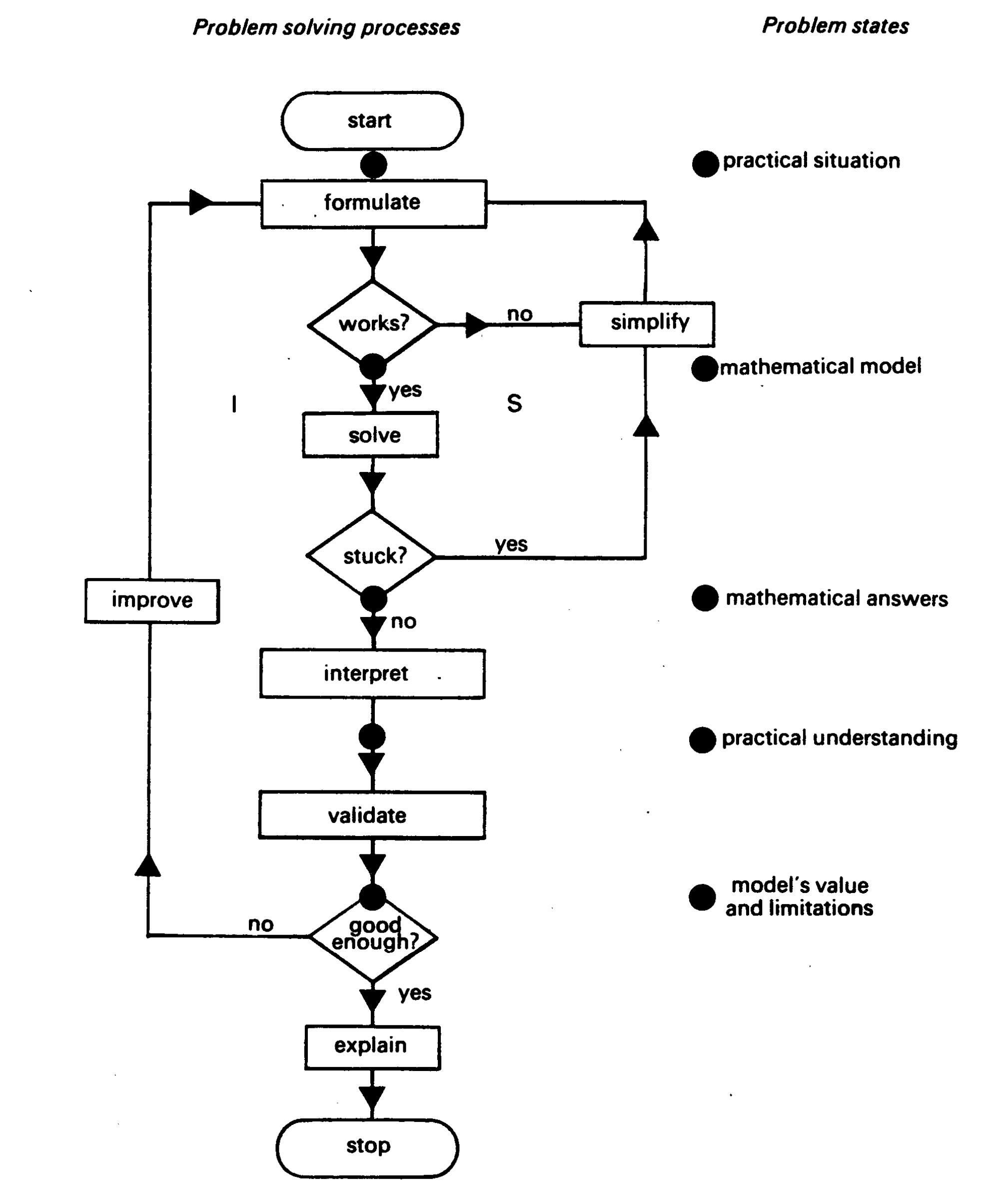

Figure 1

2. Current mathematics curricula

Everyone has been modelling with mathematics from an early age. Children estimate the amount of food in their dish, comparing it with their siblings' portions. They measure their growth by marking their height on a wall. They count to make sure they have a "fair" number of sweets. So school mathematical education has much to build on, should it choose to do so.

This informal modelling continues as people grow older. They learn to check their money before going into a shop, and check their change. As adults, they may plan their finances and the layout of furniture when they move house.

There are examples of children spontaneously using their school mathematics in more sophisticated ways. Figure 2 (from Burkhardt 1981) was produced by one 8-year old to show the 'spending money' of children in his class – and to persuade his parents that *he* should have more. He succeeded.

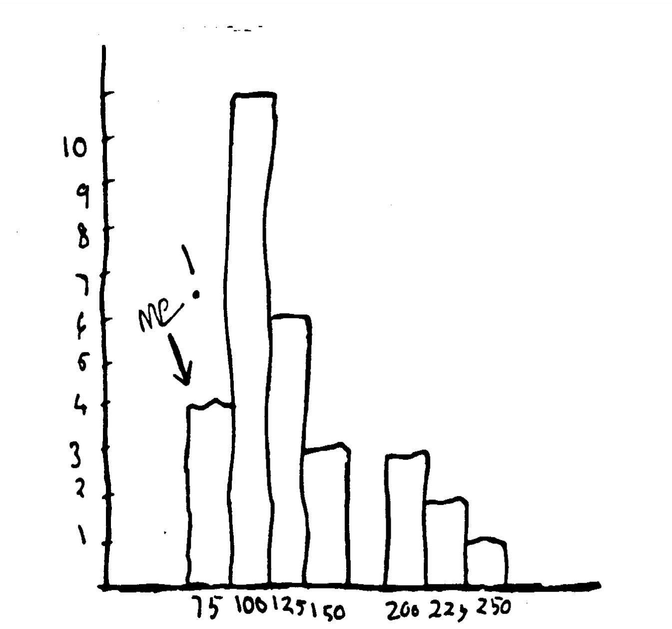

Figure 2

But, like Moliere's *Bourgeois Gentilhomme*, who was surprised to learn he had been talking "prose" all his life, people do not recognise this activity as modelling – indeed the term is still not widely used, even by mathematics teachers. (In everyday life, "modelling" usually implies the 'catwalk', not the calculator; in early school it is usually creative work with clay.) This lack of awareness might not matter if school education built on this informal foundation, continuing to develop students' ability to use their mathematics for understanding and solving practical problems of concern to them. However, most school mathematics curricula fail to deliver this or, with older students, even to address it. If you doubt this, ask some adults who are not in jobs that use mathematics professionally:

When did you last use some mathematics that you were first taught in secondary school?

Why is this? There are various factors:

- **Deferred gratification** Some argue that you must learn a lot of mathematics before you can use any; this is simply *not true* – young children, before they reach school, use counting effectively in tackling practical problems. Constantly delayed payoff is, of course, demotivating.
- **Imitative learning** Most school mathematics curricula are fundamentally imitative – students are only asked to tackle tasks that are closely similar to those they have been shown exactly how to do. This is no preparation for practical problem solving or, indeed, non-routine problem solving in pure mathematics or any other field; in life and work, you meet new situations so you need to learn how to handle problems that are *not* just like those you have tackled before.
- **Inward-looking mathematics curricula** Curriculum design in mathematics is mainly driven by people whose core interest is in mathematics itself, not in its use. Where applications are introduced, their purpose is to illustrate and reinforce the mathematical concepts and skills being taught. Such *illustrative applications* are important but they are not enough to enable people to use their mathematics autonomously to tackle practical problems as they arise; for this they need to model practical situations, choosing and using appropriate mathematics from their whole mathematical toolkit – not only the topic they are currently being taught.

  In contrast, 'own language' teaching is *outward-looking* – students learn to read, comprehend and write in many genres, from poems and stories to reasoned arguments and job applications. Mathematics education should learn from this.

The reason that mathematics has such a large proportion of curriculum time, historically and to this day, is its perceived utility in solving problems from outside mathematics[4]. Historically, it was enough for mathematics to equip people with the routine skills of calculation for bookkeeping, surveying, and like occupations; these skills would give them a lifetime of employment. Nowadays, all these basic skills can be bought as IT devices for $100 or less; a mathematics education costs ~$10,000, which is hardly a good investment for this purpose alone. Further, there is now a need for a much wider

[4] Why does mathematics have much more curriculum time than, say, music?
Is it more intellectually demanding?
Does it give more satisfaction to people?
Hardly.

range of mathematical thinking. For this the higher-level skills that develop the power and flexibility to tackle new problems is crucial because, in a rapidly changing world, no school curriculum could cover what will be needed.

In summary, there is no point in educating human *automata;* they are losing their jobs all over the world. Society now needs *thinkers*, who can use their mathematics for their own and for society's purposes. Mathematics education needs to focus on developing these capabilities.

3. Learning to model with mathematics

What kinds of learning activity do students need to build their ability to use their mathematics in tackling problems from the real world? There have been three major influences in answering this question through the design of modelling elements in curricula:

professional applied mathematicians in various fields, notably physics, who became interested in the teaching of mathematics in colleges and in schools; they often worked with teachers in professional development[5];

work on teaching heuristic strategies for non-routine problem solving, in pure mathematics (Polya 1945, Schoenfeld 1985), in the Artificial Intelligence community (see e.g. Newell and Simon 1972), and in modelling itself.

political pressure for schools to produce mathematically literate adults. (Quantitative Literacy, Functional Mathematics, and Numeracy are among the other terms used; Steen 2002, Burkhardt, Muller et al 2006 discuss these developments)

From these influences, the following types of student learning activity have been in the frame from the beginning:

modelling experience in tackling a range of practical problems using mathematics, without prior teaching on closely similar practical situations – ie problems involving greater transfer distance[6];

instruction on strategies for modelling, a set of heuristics built around variants of Figure 1;

analytical discussion by students of alternative approaches to a problem, and reflection on the processes involved.

These ingredients still remain central. How to engineer them into effective curricula is the still-unfinished story of the last 40 years. By now there is a well-developed understanding of the key role of modelling and applications in a balanced mathematical education, and some high-quality exemplification of how this can be realised in practice. To get a view of the progress, compare, for example, the well-developed examples reported through the sequence of ICTMA books (ICTMA 1982-) with the unrealised vision of Burkhardt (1981).

It is instructive in looking ahead to review the development so far. The ICTMA books and the recent ICMI Study 10 (Blum et al. 2006) give a broad perspective. They also show that, in many cases, the teaching of active *modelling* was more-or-less overwhelmed by the teaching of *models* – applications that the students were asked to learn rather than formulate themselves.

1960-80 Explorations

This period was characterised by tentative explorations of teaching modelling in both England and the US, partly stimulated by the worldwide movement for reforming mathematics education as a whole. Much of this work was individual rather than a coherent program and, though instructive, was not published. To give something of the flavour of this period, I shall again give two views, anglo-centric but compact, that serve to bring out the general points we need.

HB: My involvement in teaching modelling happened essentially by accident. I had an "Emperor's New Clothes" moment in an applied mathematics course for serving high school teachers that I had been asked to lead at Birmingham University in 1962-63. It was

[5] The UK is unusual in having a strong element of applied mathematics in secondary school curricula, essentially due to Newton and largely unchanged since. It is based on a set of models of standard situations in Newtonian Mechanics; it does not include non-routine problems or, therefore, active modelling.

[6] *Transfer distance* is a measure of how different two problems are; however, the interesting but challenging problem of inventing a robust practical measure of transfer distance has not been seriously tackled, let alone solved.

meant to complement a course for teachers by the pure mathematicians Peter Hilton and Brian Griffiths, then very active in school curriculum reform. That first year my colleagues and I taught a fairly conventional review of the Newtonian Mechanics that formed about half of the A-Level[7] mathematics course for 16-18 year old students. As the course went on, I became increasingly concerned that the standard approach lacked a vital element – explicit consideration of the modelling involved. In the following year, I included a modelling component with a much wider range of problems including, for example:

On Owning a Used Car
How old a car should you buy, and when should you sell it?

This interested the teachers, and later the undergraduates to whom I taught modelling. They explored the various costs – depreciation, repairs, fuel, etc – and chose to look, graphically, for a *minimum total cost per year*. The Newtonian Mechanics in the course now looked at the consequences of alternative assumptions to those behind the standard models: ladders leaning against walls; projectiles; weights on strings over pulleys; and so on. Over subsequent years, with both teachers and undergraduates, I developed a set of rich problems, some as open as

How do we make friends?

From 1965 on, David Wishart and I developed an undergraduate course on *Broad Spectrum Applied Mathematics* (Burkhardt and Wishart 1967). It includes both standard models and active modelling in various fields - physics, population dynamics, economics, industrial control, game theory, arms races, as well as everyday problems. This course enabled mathematics students to make an informed choice among final year courses in other academic departments. As often happens with new enterprises, the early cohorts of students did outstandingly well, and in all subjects. Also characteristically, students said that they that were spending much more time on this course than on others!

In the 1960s, there was a wider momentum in England to make undergraduate curricula less imitative, particularly through substantial investigative projects in the final year. These began in computer science, flourished in statistics and applied mathematics, then moved into pure mathematics. Ron McLone introduced modelling projects into third year undergraduate mathematics at Southampton University (see also Andrews and McLone 1976).

In the 1970's at Nottingham University, there was a sequence of developments. George Hall developed a pioneering course called 'Information Theory' – a few weeks of lectures on this topic from a modelling perspective led into a major modelling project on a topic of the student's choice. The topics chosen were varied, the quality of the student reports high, in some cases outstanding. When I moved to Nottingham in 1976, I started teaching modelling as the first unit in the basic undergraduate course called Differential Equations, starting with *On walking in the rain*.

The characteristic of these courses is that they came and went with the innovators. University faculty in UK universities have considerable freedom in what they teach; this makes experiment easy, but institutionalisation of successful innovations difficult, and rare. The breakthrough came from a different source – the UK polytechnics.

HOP: The launch of Sputnik in 1956 gave rise to national concern about the state of US math and science education. When in the winter of 1957-58 Ed Begle and Al Tucker started the School Mathematics Study Group (see SMSG 1958-72) as part of the growing "New Math" movement, they both knew me and decided to invite me to the first summer writing session. I found writing school mathematics so that it made sense surprisingly challenging!

For the next 25 years, I spent 10 to 15% of my time on math education. Bell Labs didn't object - they even promoted me, twice, so I guess it was OK by them. It got to where I was Director of Mathematical Research at Bell Labs, and Chairman of SMSG's Advisory Board at the

[7] In England, specialisation begins early – at age 16. Students normally choose just three A-Level subjects, each of which takes a quarter of their time for the last two years of high school. Many students who choose humanities take no mathematics after age 16. No fewer than seven UK government commissions since 1945 have each recommended moving to a broader curriculum (more like the Abitur or baccalaureate) – but the situation remains much the same today.

same time. The only trouble was that I had to wear these two distinct hats on the same head. How in the world was mathematical modeling at Bell Labs consistent with trying to present school mathematics in an understandable fashion? SMSG was too pure for lots of critics[8], and yet I thought it was doing the right thing, and yet I thought my job of doing and fostering mathematical modeling was doing the right thing! Over time I came to see that these can be reconciled:

> *In mathematics* one emphasises understanding of when and how and why the mathematics works.
>
> *In applications,* one correspondingly insists on understanding the real situation being analyzed, and on understanding the process of getting from the real situation to the mathematics that – one hopes – become a useful model.

I also knew that there was an awful lot more to applications than mathematical physics, circuit theory and the like. Two of the largest areas of mathematics in the Bell Labs Mathematics Research organization were not taught to engineers, or to almost anybody else, in our universities. They were, ones which had enormous usefulness to a lot of parts of the telephone business: Discrete Mathematics, and Exploratory Data Analysis. It is interesting that some data analysis techniques which I saw being invented at the research level when I first came to Bell Labs (for example, box plots and stem-and-leaf plots) are now being taught in the elementary school! And, boy, do they ever help you to model the real world!

In the first major development of modelling curriculum, Earle Lomon, an MIT Physics Professor who, like me, had been at the 1963 Cambridge Conference on School Mathematics, turned a vision into *Unified Sciences and Mathematics in Elementary School* (USMES 1969) It was developed, with funding from the National Science Foundation, by a team at EDC that he led. I was on the USMES board. For students from ages 8-11, it supported a series of extended class projects built around open problems on subjects like *Planning the footpaths in your district*, or *Welcoming a new family to the neighbourhood*. The children analyzed the situation, developed whatever mathematics, science and social science, was needed, and ended with a recommendation for action by appropriate authority. A great deal of modelling was involved, though it was not called that. Student response, both in learning and motivation, was outstanding, but USMES proved very demanding on teachers. (A study of the background of USMES teachers found that "drop-out Art teachers" handled USMES best – mathematics teachers were among the worst)

An amusing sidelight: USMES, as part of its total curriculum packages, developed "How to.." cards, outlining new and potentially useful mathematics concepts and skills. They needed new simple data analysis techniques, and I persuaded a couple of my Bell colleagues to write something for USMES for this purpose – a first step in moving Exploratory Data Analysis into education?

The strategic design of USMES reflects a general point. It was able to achieve impact by the ingenious observation that there were no fixed expectations for elementary school science in the USA. Nobody knew what elementary science really meant, and no secondary science relied on anything having happened before. So there was a niche there to be occupied, at least temporarily. We wanted to progress into junior high school; there were excellent reasons for going in that direction, but it never got off the ground. The ownership of time at the secondary level was far too rigid. You couldn't get time from math, or from conventional science, or from social studies, or from English (all of which were seriously, though not equally, involved in solving USMES problems).

Following USMES I was associated in various ways with a series of modelling developments. The second round of SMSG had a chapter on modelling but, coming as the reaction to the "New Math" began, it did not have much impact.

I feel I was one of the creators of a later project – *Mathematics: Modeling our World* (COMAP 1997-8), the high school textbook series developed by the ARISE project in the 1990s. It is very much a realization of my dreams for modeling in the schools, but well quenched with a bucketful of cold water reality of what one could actually do. (One of my favorite ideas

[8] e.g. (SMSG 1962) begins elementary school with sets.

was the development of logarithms from the attempt to model the notion of information. It was moved from grade 9 to 10 to 11 to 12 to "You can do it in college sometime.". I now teach it at Teachers College as part of a course on applications of mathematics in engineering and science.) The ARISE curriculum is challenging to teachers, both in the skills it demands and its outward-looking attitude to mathematics. It is not widely used but it is valuable in exemplifying a way of building mathematics around applications.

These two sketches aim to give something of the flavour, and excitements, of the period. Other explorations of how modelling might be introduced into school mathematics were happening in various forms and places around the world, notably Australia and the Netherlands. The growth of computer education in schools was another lively area – students were writing programs, often in BASIC, that were explicit models of practical situations but, again, there was little explicit focus on the modelling process.

1980-2000 Establishing exemplar courses

From 1980 onwards, the international movement for the teaching of modelling was established, and steadily gained momentum. A rich pattern of developments gradually emerged. The story can be followed in more detail in the books (ICTMA 1982-) arising from the ICTMA conferences, a continuing biennial series which David Burghes launched at Exeter in 1981. These played a central role in establishing communication among the international community of innovators in the teaching of modelling, and applications in general. Here we can mention only a few developments that still seem important.

The annual *Undergraduate Mathematics Teaching Conferences* (UMTC 1976-) brought together each year innovators in teaching mathematics from across UK higher education to share their experience and, often, frustrations at the conservatism of their home departments. Modelling became a major theme in these conferences.

In the UK during the 1980s some polytechnics established modelling courses. This development is worth describing since it has lessons for the still-unsolved problem of establishing an innovation like modelling as a long-term element in the curriculum. Polytechnics did not award their own degrees but those of the *Council for National Academic Awards* (CNAA), which had to approve all courses. The CNAA Mathematics Panel supported modelling initiatives and later, crucially, decided that *all polytechnic undergraduate courses in mathematics must have a modelling component in all three years*. This institutionalised modelling in this sector in a way that, as far as I know, has not yet happened in mathematics curricula elsewhere. The ICTMA books illustrate the major contribution to the development of the teaching of modelling that people in these institutions have made. The polytechnics have since become universities, and now have the freedom to design their own courses. So far, modelling in mathematics seems alive and well. In contrast, modelling is not part of the mathematics course at many other universities – including Nottingham where much of the exploratory development took place.

There were important developments in this period in the US at high school level (Pollak 2003 gives more detail). COMAP, which had produced the UMAP sequence of undergraduate applications modules, developed *For All Practical Purposes* (COMAP 1988), a problem-based set of modelling examples from a wide range everyday life and professional fields. Among many memorable examples is one on alternative voting systems – each of five systems gives a different winner from the same set of preference choices.

Building on this, NSF funded the ARISE project, referred to above, as one of a set of mathematics curricula based on the NCTM Standards. The goal here was to teach a lot of new mathematics, particularly for modern fields of application, so standard models are used as illustrative applications – but there are also regular opportunities for active modelling by students.

Meanwhile in England, the Shell Centre team was inspired by USMES to develop an integrated scheme of teaching and assessment based on modelling projects. We sought to reduce the considerable challenge to teachers through an unusual and carefully engineered design that gave teachers more explicit support without detailed guidance to the students on particular solution paths. Working with the Joint Matriculation Board, a major examination provider, the team developed five three-week modules of teaching and assessment on *Numeracy through Problem*

Solving[9] (Shell Centre 1987-89). The style and range of problems is indicated by the titles: *Design a Board Game, Produce a Quiz Show, Plan a Trip, Be a Paper Engineer,* and *Be a Shrewd Chooser*. As well as embedded individual assessment of *Basic* level performance, there were final examinations at two levels, *Standard* and *Extension*, which assessed the student's ability to *transfer* their understanding to less- and more-distant situations respectively. The design team was led by Malcolm Swan, who had earlier designed *The Language of Functions and Graphs* (Swan et al 1986), which pioneered the introduction to UK curricula and examinations of these modelling techniques.

Numeracy through Problem Solving was later adapted as an alternative "syllabus" in the Mathematics GCSE, the standard high-stakes examination for 16-year olds. As so often with innovations, it disappeared later almost by accident – as an incidental consequence of a general government decision to standardise syllabuses.

2000+ Large scale impact? Where are we now? A variety of 'proof of concept' courses, some outlined above, have been developed. However, the large-scale impact of modelling on mathematics curricula at every level remains modest at best. The remainder of this paper is concerned with why this is, and what we might do to move forward.

4. Making modelling a reality in mainstream curricula

The problem of making modelling, or any substantial innovation, a reality in every classroom is far from solved, even when there is support at policy level. The challenge is always underestimated by both governments and the professional leadership; they tend to assume that once "difficult decisions" have been taken, implementation is straightforward. This is far from true – classroom outcomes in line with the goals are very difficult to achieve. This should not be surprising, since such innovations usually demand profound changes in the well-grooved day-by-day professional practice of many people. (Typically, about 1% of the population in developed countries teach mathematics) In this and the following sections, we will review the challenges to the large-scale introduction of modelling and how they may be tackled. History, as outlined above, suggests that achieving substantial large-scale change will not be easy – and that underestimating the challenge will guarantee failure.

Some hopeful signs

But first, I would like to note some positive features of the current situation, which will certainly help progress in the future:

- **Mathematical Literacy** is now a new, stronger focus of attention at policy level in many countries, aimed at ensuring that school mathematics is functional at a practical level for all adults.
- **PISA** The OECD *Program for International Student Assessment* in Mathematics (PISA 1999, 2003) is focussed on mathematical literacy, with items in the tests that all seek to reflect problems in the real world. They have a significant modelling demand. The results are now getting political attention, at least comparable to the 'purer' TIMSS tests, ensuring that students' ability to model with mathematics will be important at policy level.
- **Information Technology**, spreading slowly in classrooms, removes much of the drudgery from modelling reality – long calculations, collecting and handling data, etc. It is also at the core of both mathematics and modelling in the real world. However, introducing IT presents major challenges to school systems, outlined below.

What are the tough challenges that offset these hopeful signs? We will now briefly review the main ones.

Learning and teaching mathematical modelling

Teaching modelling needs a wider range of teaching strategies than most teachers use in delivering the essentially-imitative curriculum that dominates classrooms in most countries. Why? What additional skills does a teacher need to acquire in order to enable their students to learn to

[9] *Numeracy* was originally defined in a British Government Report (Crowther 1959) as "the mathematical equivalent of literacy". It is now often used to mean just "skill in arithmetic".

model with mathematics? How do we get curricula that develop them? We address these key questions in turn.

Skills beyond 'basic' As we have seen, modelling involves all the key aspects of 'doing mathematics' which may be summarised (see e.g. Schoenfeld 1992) as:

knowledge of concepts and skills

strategies and tactics for modelling with this knowledge

metacognitive control of one's problem solving processes

disposition to *think* mathematically, based on **beliefs** about maths as a powerful 'toolkit' (rather than just a body of knowledge to be learnt).

These are not, of course, independent elements but must be integrated into coherent **modelling practices** for tackling whatever problem is at hand.

Richer learning activities For learning these, the main classroom elements that are seldom found in traditional curricula are:

active modelling with mathematics of non-routine practical situations;

diverse types of task, in class and for assessment;

students taking responsibility for their own reasoning, and its correctness;

classroom discussion in depth of alternative approaches and results;

teachers with the skills needed to handle these activities.

These imply a profound change in the *classroom contract*, the set of mutual expectations between teacher and students as to their respective roles and actions (see Brousseau, 1997). Table 1 (Burkhardt et al, 1988) illustrates the necessary role changes:

for imitative learning	for modelling, add
Directive roles	**Facilitative roles**
Manager	Counsellor
Explainer	Fellow student
Task setter	Resource
with students as	*with students as*
Imitator	*Investigator*
Responder	*Manager*
	Explainer

Table 1. Teacher and *Student* Roles

Broader teaching strategies What extra skills do teachers need to make this a reality? The key elements here include:

- **handling discussion** in the class in a *non-directive* but supportive way (see e.g. Swan et al. 1986, inside back cover), so that students feel responsible for deciding on the correctness of their and others' reasoning and do not to expect either answers or confirmation from the teacher;
- **giving students time and confidence** to explore each problem thoroughly, offering help only when the student has tried, and exhausted, various approaches (rather than intervening at the first signs of difficulty);
- **providing strategic guidance** and support without structuring the problem for the student or giving detailed suggestions (see e.g Shell Centre 1984, inside back cover);
- **finding supplementary questions** that build on each student's progress and lead them to go further.

This is challenging at first, but teachers who acquire these skills continue to use them. Well-engineered materials can provide enormous support to teachers and students who are engaging in learning modelling. Such materials are essential for most teachers in their first few years of such teaching, if they are to succeed.

It is not surprising, given this brief summary of the key elements in learning and teaching modelling, that it has been difficult to achieve in most classrooms. However, these are probably not the core reasons, which are systemic; we shall discuss them in the next Section. But first, brief comments on two issues of importance for curriculum design – mathematics and technology.

What mathematics do students need?

Almost the whole of pure mathematics, however apparently abstruse, has proved useful for solving problems in some field of practical importance. The use of prime number theory in encryption of internet messages is but one of a host of wellknown examples. Nonetheless, students cannot learn everything at once so a priority order is helpful. It should reflect accessible power over real world problems rather than the priorities of mathematics itself, when these are different. Here a few examples must suffice.

Some neglected topics are particularly powerful; many, including spreadsheets and programming, are linked to technology. This reflects the practical importance of the *discrete mathematics* of numbers, compared to the *continuous mathematics* of quantities, which currently dominates curricula beyond the elementary school. Algebra remains the key to higher performance, in modelling as in so much mathematics; however, aspects that are crucial for modelling, particularly the *formulation* of algebraic models, are hardly touched in many current curricula, which focus elsewhere – mainly on solving given equations. Geometry, too, needs a change of emphasis for modelling – with more emphasis, for example, on design. *Statistics and probability* are essential in thinking sensibly about many practical problems.

Technology – core and curse

On the one hand, technology is at the heart of the modelling enterprise. Hardly anyone does any modelling, on however simple a topic, without using some technology, if only a calculator. Spreadsheets provide invaluable support for setting out analytic structures, exploring changes in assumptions and values, checking calculations by alternative routes, and so on. Any modelling course that does not use such resources is out of touch with the real world – not uncommon in school curricula, of course, but crucial here.

On the other hand, technology presents great problems for large-scale implementation of curriculum and assessment, largely because of the mismatch of timescales between technological change (~2 years) and educational change (~10 years). This produces variations in time and space of hardware and software provision, and in teacher familiarity with its capabilities. One can achieve a lot with a simple calculator; one can achieve far more with more sophisticated current technology, provided one has absorbed its capabilities - and developed a curriculum that will enable teachers to realise them in the classroom. This presents a dilemma for school systems.

Various compromises have been tried; all have problems. Limiting the technology to what is *universally available* holds back progress, of individuals and of the system. On the other hand, assuming that *all useful resources* will be available will disadvantage the many who don't have them. A wide range of provision presents huge problems in selecting the tasks that students can tackle at different levels; only very open tasks, or those where technology doesn't help, qualify. This is too restricting.

This dilemma is most acute in the design of high-stakes examinations, where 'fairness' is understandably seen as essential. Further, in systems with such examinations, the implemented curriculum in most classrooms is largely focussed on what is tested, so the issue must be faced.

A modular approach, where there are parallel modules of curriculum and assessment built around *different* levels of technology provision can work well – for example, algebra with graphing calculators, with spreadsheets, with programming (or with none of these). But this brings even more complexity to the already-challenging change process.

These problems seem likely to remain with us, but further imaginative work, backed by well-engineered materials for teaching and assessment will help, at least, to mitigate the difficulties.

On a more optimistic note, equipping every student with a wireless laptop computer of their own, for use at all times, could provide a plateau of provision that is adequate for most things in school. It could also revolutionise teaching and learning, though the curriculum development challenge will be enormous. It is an irony that the "one-laptop-one-person" initiative (OLPC 2005), intended for the developing world, may provide this universal "$100 laptop" fairly soon – but only in prosperous countries.

5. How do we get there?

The need for students to learn to model with mathematics is widely accepted. There are now well-engineered exemplars of how this can be achieved in schools. Why then do we not have modelling as an integral part of mathematics curricula worldwide? This key problem requires a discussion of education systems and the dynamics of the change process within them. Though there is, of course, variation between systems, many of the major challenges to large-scale implementation of innovations are common.

Barriers and levers

I will now look at some of systemic barriers to large-scale implementation of modelling. These four examples, though important, are mainly to illustrate the mode of analysis. Discussions with 'change agents' who are working to forward reform programs of all kinds reveal both the barriers they face and, with a few, some successful experience in tackling them. Successful approaches can then be developed so as to be useful to, and usable by, others. Such discussions also help to motivate the development of further useful tools that support reform.

Barrier 1: Systemic inertia

To put in perspective the limited large-scale implementation of modelling, we should note that it has proved difficult in most countries to establish *any* profound innovation in the mainstream mathematics curriculum. Compared with, say, a home or a hospital, the pattern of teaching and learning activities in the mathematics classrooms we observe today is remarkably similar to that we, and even our grandparents, experienced as children. The EEE style of teaching (*Explanation*, worked *Examples*, imitative *Exercises*) still dominates, as does the focus on learnt facts, concepts and skills. Learning activities that require less directive teaching styles, notably "higher-level thinking" in all its forms including modelling, remain rare, along with the increased student autonomy implied.

This is an important observation because it is easy to underestimate the challenge of large-scale implementation – to assume, reasonably it seems, that, if something is recognised as "good" and "important", it will naturally be taken up as part of the mainstream curriculum.

Rather we should recognise that we need to change:

- **habits** – the well-grooved day-to-day practice of large numbers of professionals;
- **beliefs** – the expectations of parents, politicians the public, and mathematics teachers about what mathematics *is*, usually based on their own school experience;
- **teaching skills** – the skill and knowledge base of teachers and teacher educators, outlined above;
- **power balance** – within the subject peer group between "basic skills" v "problem solving", pure v applied, etc;

It is understandable that that will require a *powerful and coherent combination of pressure and support*.

Among the arguments used to resist the introduction of new elements like modelling, "There is no time" is common, powerful and, in its literal sense, valid. In fact, many mathematics texts already contain far more material than can possibly be covered in the time available[10]. While much time is currently wasted in repetition of exercises and revision of topics, that point is rarely persuasive on its own; any alternative must be explicitly shown to work in practice, which requires systematic development and evaluation.

It is not always difficult to identify explicit elements for elimination. In one case, we found that "algebraic inequalities" were happily eliminated from an examination syllabus for 16-year olds. The teachers on the committee were unanimous that "We teach it but they can never do it"; others agreed that this topic is not essential. The purpose was to liberate 3 weeks teaching time, and one examination question, for a modelling component. The general lesson that emerged is the need to be explicit in what is to be cut and how much time is freed for new material.

Among the **key levers** for tackling resistance to change are *curriculum descriptions*, supported by *well-engineered materials* to support *assessment, teaching, professional development* and *public relations* (in the literal sense) that are well-aligned with the each other – and have been shown to work well in realistic circumstances of personnel and support. Unless these tools and processes for their use are in place, modelling experiments, however successful in themselves, are likely to be evanescent, distorting back to the traditional and fading away with those who led them. These things, and the process for improving their quality, will be discussed further below.

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[10] This is particularly true in the US where "adoption committees" decide which books can be bought with state funds. Their list of requirements rarely takes account of time considerations so, to ensure that all the requirements are met, the books often contain three times as much material as can be taught in the time. The selection is then made by teachers who, naturally, tend to choose what is familiar.

Barrier 2: The real world

The real world is an unwelcome complication in many mathematics classrooms. The clean abstraction of mathematics is something that attracted many to teach mathematics, particularly at the higher levels. Teaching mathematics, they say, is demanding enough without the messiness of modelling reality. Further, the public, based on its own school experience, shares these beliefs about "proper mathematics". Modifying their views is a necessary condition for progress. What **levers** have we?

Illustrative applications have long been used to provide both concrete embodiments of a new concept or skill, and illustrations of how it is used (there is some acceptance that utility is important). These are essential, but on their own not nearly enough – the problems will arrive pre-modelled.

Another key lever with both teachers and students is *student interest and motivation*. Most students find modelling courses more relevant to their lives than the mathematics they are used to – and more interesting. Where well done, modelling is also more fun. The performance across mathematics of some low-achieving students shows really substantial improvement – many have been 'turned off' mathematics by its perceived irrelevance to anything that interests them. Teachers find such evidence persuasive, particularly from colleagues they know. However, such findings also need a firmer research base and good presentation, in general and in the context of each particular modelling course (see Barrier 4). Some pure mathematicians will still find this unconvincing, indeed threatening. They protest "This will undermine the basics", ignoring the evidence that "back to basics" approaches actually lead to lower scores, even in the narrow tests usually used. For them and the public at large we need other levers.

Building public understanding of modelling and its central role in providing students with important skills is essential to progress. It requires *public relations tools* built on the specific local changes for use in parents meetings at school, and with the community through the media, as well as with decision takers. Here a small number of change agents in each community (of whatever size) can play a key role, as long as they have good tools to help them explain and exemplify. (As we saw above, modelling became a core part of mathematics in the English polytechnics through influencing decisions of the then controlling body, CNAA.)

Barrier 3: Limited professional development

In many countries teachers are expected to deliver a curriculum on the basis of the skills they acquired in their pre-service education, consolidated in the early years of classroom practice. This approach may have worked well when the curriculum changed little during a teacher's career; it is clearly inadequate now. However, with notable exceptions like Japan, continuing professional development is not yet an integral part of teachers' day-to-day work. Elsewhere opportunities for professional development are often occasional and/or voluntary, taken up by a minority of teachers, usually those who need them least.

Further, professional development programs are rarely developed with the kind of rigour and imagination that is expected in the development of assessment or teaching materials – or of products in other fields like medicine or engineering. For example, it is rare that there is any effort to see whether *teachers' classroom behaviour* changes after taking part in the professional development program. Where this has been done, changes have often been *undetectable*. This is not surprising, since the feedback with which such programs are evaluated and developed is focussed on whether the participants found the experience *interesting and useful* – not the same thing. Indeed many regard such research as intrusive and inappropriate; they see professional development as a civilised exchange between fellow professionals, who take whatever they find useful from the experience. Yet knowing the classroom outcomes are surely crucial.

This craft-based 'professional' approach ignores the obvious fact that, as with all skilled activities, teachers vary enormously in their range of skill in teaching, and that none of us are as aware of our limitations as we should be. It is important to recognize these distinctions. if appropriate support is to be provided. I find it useful to distinguish *virtuosi, craftsmen, labourers* and *incompetents*. Too much attention has been misdirected – by professional leaders on virtuosi, and by politicians on incompetents; since both groups are relatively small, *system* performance depends on turning labourers into craftsmen, and helping these to continue to develop.

Levers are implicit in the above. *Time* for all teachers, mainly as part of their working week, to work together, supported by tractable *professional development materials and programs with specific goals* which have been *systematically developed* and shown to *enable people like themselves to achieve these goals in their own classrooms*. The relevance of this for teaching modelling is clear but many of the same teaching tactics and skills help achieve other widely accepted curriculum goals that involve non-routine performances – for example, problem solving in pure mathematics. They also support the long-term learning of concepts and skills.

These last assertions need to be further researched and documented in a way that will be convincing to the proper audience for research – the positive thinking sceptic who advises policy makers. This brings us to Barrier 4.

Barrier 4: The role and nature of research and development in education

Research and development in education, as compared with other applied fields, is not well organised for turning research insights into improved practice. Burkhardt and Schoenfeld (2003) looked at this process, and how it can be improved. Here is the essence of the argument.

In education there is an amalgam of three different research traditions – *humanities*, *science* and *engineering*, which are those characteristic of those different groups of departments within a typical university. While the education research community favours empirical insight-focused *science* research, policy is still largely dominated by the *humanities* tradition of critical commentary with no empirical testing[11]. While good insight-focused research of either kind identifies problems and suggests possibilities for progress, it does not itself generate reliable solutions that can be directly implemented on a large scale. To achieve that, research-based development and robust well-tested models of large-scale change are both essential. This impact-focused *engineering research* tradition, with products rather than just papers as its key outputs, is still peripheral in education. (See Burkhardt 2006 and Schoenfeld 2002, for more on these ideas.)

In successful research-based fields of practice (e.g., medicine or the design and engineering of consumer electronics or communication devices) one finds:

A. Well established mechanisms for taking ideas from laboratory scale to widely used practice. Such mechanisms typically involve multiple inputs from established research, the imaginative design of prototypes, refinement through feedback from systematic development, and marketing mechanisms that rely in part on respected third-party in-depth evaluations.

B. Norms for research methods and reporting that are rigorous and consistent, resulting in a set of insights and/or prototype tools on which designers can rely. The goal, achieved in other fields, is cumulativity – a growing core of results, developed through studies that build on previous work, which are accepted by both the research community and the public as reliable and non-controversial within a well-defined range of circumstances.

C. A reasonably stable theoretical base, with a minimum of faddishness and a clear view of the reliable range of each aspect of the theory.

D. Teams of adequate size to grapple with large tasks, over the relatively long time scales required for sound work of major importance.

E. Sustained funding to support the research-to-practice process on realistic time scales. In other fields, substantial funding followed the public perception of practical impact.

F. Individual and group accountability for ideas and products – do they work as claimed, in the range of circumstances claimed?

In education, none of these is reliably in place, though the development of *design research* is a move in this direction. The emphasis in educational research remains on small, neat studies of unavailable treatments that produce interesting insights of unknown generality; these are not an adequate basis for design or evaluation.

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[11] This dominance of "plausible argument" also warrants policy-makers reliance on their own "common-sense" views, in a way that would be unimaginable in other fields like, say, medicine, where research is taken more seriously.

The international modelling community is equipped to move things forward. It has a body of successful exemplars that provide proof-of-concept evidence that modelling can be made a practical large-scale classroom reality. It has the communication structure with which to share expertise, and build the case for the government support that will be essential to the further well-engineered research and development that will be needed before large-scale introduction. The opportunity is there but the challenges remain to be overcome.

Other barriers and levers

These four are some of the important barriers that are regularly reported by people who have sought to introduce modelling, or other innovations involving higher-level strategic skills, into mathematics curricula. Since the dynamics of curriculum change are not yet well understood, other barriers need to be recognised and clarified, and ways of overcoming them developed.

The development of a broader range of tools for advancing the system changes that curriculum improvement requires is a substantial challenge that will require new design and development skills. The traditional range of accepted tool types (mostly teaching materials and tests) is slowly being extended, as well as improved in quality. Materials that enable less experienced leaders to deliver more effective professional development is one growth area.

The US National Science Foundation has funded the development of a *Toolkit for Change Agents* (http://toolkitforchange.org/) that looks holistically at the problem outlined in this section. What barriers do change agents come up against? How have others overcome similar barriers? What tools can help? Where can they be found? If they do not exist, can they been specified and developed? This Toolkit is designed to find examples of effective practice, and make the strategies and tools involved available to others in a form they can use. This work is ongoing.

Making progress

The barriers to reform outlined in this section are formidable indeed. However, the levers for forwarding improvements such as the teaching of modelling can be powerful. Frontal assaults may not be the most effective way; progress is likely to depend on identifying and seizing opportunities for system change as they appear (as USMES identified elementary school science as an opportunity). The current political concern for mathematical literacy presents both an opportunity and a substantial challenge. We will see what emerges around the globe.

6. The roles of assessment

Finally, as an example of a set of tools for promoting change, I want to look a little more deeply at one key area, both of challenge and of opportunity – the high-stakes assessment for accountability that, in some countries, dominates the world of teachers and students. Many innovators regard assessment as, at best, a regrettable necessity, but it also offers a powerful lever for advancing improvement. Briefly, assessment provides a means of:

- **A 'measuring' performance** – assessing levels of performance of individual students in a domain, for one or more of a variety of purposes – to give formative feedback to improve teaching and learning, to guide decisions on future opportunities for the student, ….. and for teacher and school accountability to parents and government.

If the results of an assessment have significant consequences (for students, teachers, or schools), such high stakes assessment *inevitably plays two other roles*:

- **B specifying curriculum goals** – identifying to teachers and their students, through the types of task in the tests and the scoring schemes, those aspects of performance in the domain that are seen by 'society' as important; and thus
- **C driving teaching and learning** – leading teachers to concentrate on those aspects in their teaching, at the expense of important aspects of performance that may be recommended but are not tested.

These last two points were summarised long ago as *WYTIWYG* ("What You Test Is What You Get"). This represents an opportunity rather than, as so often, a threat.

Balanced Assessment is the term MARS (1995-98) coined for tests which take B and C as inevitable, and *accept the consequent design responsibility* to ensure:

Curriculum Balance Teachers who 'teach to the test', as most will, are led by the test design to provide an *implemented curriculum* in their classrooms that is reasonably balanced across all the goals of the *intended curriculum*.

Learning Value Because such assessment takes significant time, the assessment activities themselves should be useful learning activities.

Professional development activities built around high-stakes assessment are unusually powerful, in that:

- they are taken up by most teachers, not just the enthusiasts;
- sessions built around a rich task bring out the core issues – of the subject, and of learning and teaching it – in a vivid and concrete form;
- teachers at every level learn in a 'constructive' way, generalising from specific experiences, that is both deep and closely tied to classroom practice.

The development of tools to support these things will pay off in implementation, if and only if they are integrated in a systemic approach.

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