Quantitative Literacy for All: How Can We Make it Happen

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This paper traces the essential elements of QL—from performance goals, through student learning activities, to their teaching implications and those for teacher education. It takes an engineering research perspective, pointing out that the power of situated learning depends crucially on how well designed and developed the situations are. It sees QL primarily as an end in itself, and a major justification for the large slice of curriculum time that mathematics occupies. It also points out that QL can be a powerful aid to learning mathematical concepts and skills, particularly for those who are not already high achievers.

I approach this important topic from the perspective of an educational engineer. Whereas the ‘science research’ approach aims for improved *insights* into the system being studied, these are only a starting point for an engineer; ‘engineering research’, in education as elsewhere, seeks *direct impact* on the system through developing improved tools and/or processes (Burkhardt, 2006). This means that I shall say as much about models and exemplars as about principles and research questions.

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I shall treat as equivalent the various terms used for QL around the world: *quantitative literacy* or *quantitative reasoning* (US), *functional mathematics* (recent UK), *mathematical literacy* (most other places) and *numeracy* (originally defined in the British Crowther Report (1959) as "the mathematical equivalent of literacy" but now too-often corrupted to mean procedural skill in arithmetic). The distinctions between these terms that people try to make (see e.g. Smith, 2005) are minor compared with the distance of them all from current classroom reality.

In the language of situated learning, this paper looks at two 'activity systems' (Greeno, 1998), the QL classroom and the professional development environment, pre-service and in-service. There is too little space here to discuss the most intractable, the educational systems of which these are part. Though often related to design research (Brown & Campione, 1994), most analyses in terms of situated learning seem to lack the evidential warrants that provide an adequate basis for informing design (Burkhardt & Schoenfeld, 2003). They fail entirely to address the engineering research that is needed to develop products robust enough for effective large-scale use.

What is QL?

PISA (OECD, 2003), representing an international consensus, defines it thus:

> *Mathematical literacy* is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.

More succinctly, QL is *thinking with mathematics about problems in everyday life*. However, such verbal descriptions on their own are ambiguous—they are easy to re-interpret in terms of one's own experience. I find it clearest to specify performance goals through examples of assessment tasks, with their scoring rubrics, if needed. In the Appendices I offer two tests of mathematical literacy, one appropriate for "all well-educated adults," the second for students around Grade 8. QL is exemplified by the tasks in these tests, perhaps along with some PISA tasks. The notes accompanying each test on the success of those who have tried them confirm that there is work to be done to make QL a reality—no surprise to any of us. Here let us look at just one of these tasks, based on a number of UK cases:

> *Do Sudden Infant Deaths = Murder?* In the general population, about 1 baby in 8,000 dies in an unexplained "crib death." The cause or

causes are at present unknown. Three babies in one family have died. The mother is on trial for murder. A medical expert witness says:

> One crib death is a family tragedy; two is deeply suspicious; three is murder. The odds of even two deaths in one family are 64 million to 1.

Discuss the reasoning behind the expert witness' statement, noting any errors, and write an improved version to present to the jury.

Situated learning and quantitative literacy

How does this task relate to situated learning? The basic idea—that we need to understand learning as an active process in which students have to be engaged constructively—is long familiar, recognized by Dewey and other theorists and restated in the NCTM (1989) Standards as follows: "A person gathers, discovers, or creates knowledge in the course of some activity having a purpose."

At a more detailed level, Engle and Conant (2002) suggest that "productive disciplinary engagement can be fostered by designing learning environments that support:

- problematizing subject matter,
- giving students authority to address such problems,
- holding students accountable to others and to shared disciplinary norms,
- providing students with relevant resources."

Much of the research in the field amounts to rephrasing and refining these ideas, adding illustrations though often with little detail, and distinguishing this from other modes of analysis of learning. Greeno (1998) puts it thus:

> Unlike behaviorist and cognitive research, which focus primarily on individuals, situative research takes larger systems, which we can call *activity systems,* as its primary focus of analysis. An activity system usually has a few people in it, along with whatever resources in the environment that they are interacting with. The main question for the analysis is how such systems function, especially how their components are coordinated.

For this paper the bottom line on situated learning is that meaningful classroom experiences with sense-making produce engaged, empowered, effective learners—not the dominant impression of current mathematics classrooms.

To justify QL as a curriculum component, some mathematicians go further and assert:

For most learners, thinking with mathematics about problems from everyday life offers powerful support for sense making in mathematics.

However true, this is an extraordinarily inward-looking view. For me and, I believe, for most people, the *practical* utility of being able to *think mathematically about practical problems* is the prime motivation for studying mathematics; its inherent beauty and elegance are merely a welcome bonus. I will return to this issue later, after discussing the challenges of making QL a classroom reality.

Tackling real world problems

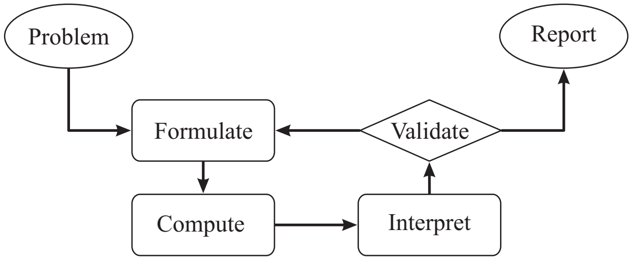


Figure 1. The phases of modeling

The standard diagram in Figure 1, summarizing the top-level processes of QL[1], makes it clear that this involves more complex thinking than the short imitative exercises that dominate mathematics classrooms. However, in the last 40 years, we have learned how to teach the higher-level skills involved (see e.g. Burkhardt with Pollak, 2006). In brief, the following types of student learning activity are necessary:

- *modeling experience* in tackling a range of practical problems using mathematics, without prior teaching on closely similar practical situations—i.e. non-routine problems involving significant transfer distance[2];
- *instruction on strategies for modeling*, a set of heuristics that facilitate the processes of Figure 1;
- *analytical discussion by students of alternative approaches* to a problem, and reflection on the processes involved.

These ingredients are central. How to engineer them into effective curricula at all levels is the still-unfinished story of the last 40 years. However, there is now a well-developed understanding of the key role of modeling and applications in a balanced mathematical education, and some high-quality exemplification of how this can be realized in practice. The recent ICMI Study

14 *Modelling and Applications in Mathematics Education* (Blum, Galbraith, Henn, & Niss, 2007) gives a wide-ranging review of the field.

Details matter: The importance of good engineering

Much of the discussion of QL in the US is, like the text above, about matters of general principle. Probably the most important contribution I can make here is to point out that general principles are not enough—that *the details matter*. The difference between Mozart, Salieri and their many contemporaries of whom we have not heard was not in their general principles—the laws of melody, harmony and counterpoint were common to them all; it lay in their use of those principles in the design and development of their music.

Good engineering recognizes this. Its essential elements are familiar from other fields:

- build on prior work, both research and practice;
- use expert and imaginative designers;
- refine the designs through an iterative process of revisions, based on feedback from trials—ultimately in circumstances of personnel and support like those of the users;
- monitor in the field for rare 'side effects.'

This is a more complicated, and expensive, process than the traditional authorship approach. We compare these approaches in 2.2 and 2.3.

Design heuristics

I say "general principles" because outstanding designers (and composers) work with sets of design strategies and tactics which are often not widely shared or, sometimes, even articulated by the designers. As in any field, many of these are heuristic strategies and tactics, derived directly from experiment—phenomenological and detailed. To take just one powerful example that is relevant to QL (see e.g. Phillips et al., 1988 for evidence): *Moving students into teacher roles leads to higher-level learning.*

Some outstanding teachers can work from general guidance alone. However, many teachers, even if they try, will fail and the whole enterprise may be discredited[3]. Excellence in design and systematic care in development of support will be important in equipping most teachers to make the innovation work in their classrooms. After all, most follow a published text in teaching mathematics they know well. Are they likely to succeed in demanding new areas with *less* support?

Because of its importance, I am going to illustrate this point in some detail, comparing the *traditional approach*, used by innovative teachers and most writers of educational materials, with the more powerful, and more expensive, *research-based* approach of good engineering. Those who produce materials for others to use face two kinds of challenge:

A. How effectively do the learning activities advance student learning?

B. How well do the materials communicate the necessary knowledge and skills needed by teacher-users to handle the activities in the classroom?

I will use the problem of *designing a board game* to exemplify the general points. This learning activity, in which students design, construct and evaluate their own games, has often been introduced by innovative teachers in British schools. By way of comparison, we look at a *Numeracy through Problem Solving* module we developed, offering both teaching materials and high-stakes assessment (Shell Centre, 1987–89).

The traditional approach

Most educational materials are written by experienced teachers who seek to show the way they work to fellow professionals. How are new learning activities developed? Let us begin with the board game example. Typically, an innovative teacher asks her students to bring to class their favorite board games. Groups of students play some of these games, thus exploring some existing examples. The teacher then asks each group to design a game of their own, providing an opportunity for creative independent thinking and a product of which they can be proud. This can be a worthwhile activity in which some mathematics may or may not emerge.

However, commercial games are remarkably well-designed and carefully developed; the class has little chance of producing something comparably good. What emerges is usually a minor variant of a commercial game, narrowing the creative horizon. In terms of Question A above, this is a valuable enrichment of any curriculum dominated by imitative exercises but a too-limited experience for the students. The *design load* on the teacher (an important concept) in this approach is large. For the teacher-author writing up their approach, without extensive trialing by others, provides no evidence to answer the communication question B.

An engineering research approach

Numeracy through Problem Solving (NTPS) grew from my concern that many students see school mathematics as irrelevant to their present or future lives —except as “something we have to take.” Earlier exploratory developments had shown that QL skills can be taught. NTPS was partly inspired by USMES (1969)—and recognition of the unreasonable demands this pioneering project made on teachers.

We started by brainstorming possible topics with a group of exceptional teachers. The 30 topics that seemed promising were looked at through further discussion and some informal trials by the teachers. Some continued to look good; others less so. (*Run a Swap Shop,* in which students bring things they want to barter, was popular but produced classroom chaos. With hindsight we should have seen that 30*x*29/2 potential trading pairs was a problem!) We reduced the 30 to 10 and, over 4 years, developed 5 modules.

The design team, led by Malcolm Swan, decided on a 4-stage strategy for the learning sequence. *Design a Board Game* was the first to be developed. It worked out like this:

Stage 1: Students explore the domain by working on and evaluating exemplars provided. For this module we designed five *bad* games. The student's job was to find the (many) faults in each and suggest improvements. For example:

The Great Horse Race Rules

1. Put the horses on their starting positions, 1 to 12.
2. Each player chooses a different horse. If there are only a few players, then each can choose two or three horses. The remaining horses are still in the race but no one 'owns' them.
3. Roll two dice and add the scores.
4. The horse with that number moves one square forward.
5. The first horse to the finish wins.

In *The Great Horse Race* every student can make progress, including many who would normally have great difficulty with the binomial distribution of probabilities. (The games were also designed to bring out mathematical concepts.) The students learned a lot from each game, notably the basics: that a game needs a board, rules for play, and for winning. (They were also delighted that the teaching materials, containing so many mistakes, came from the examination board—an unexpected bonus.)

Note that, because the games had faults, some obvious, others less so, the students' own games were going to be better than these, guaranteeing a feeling of success.

Stage 2: Generate and sift ideas, make a plan. Students in a group share ideas for various new games, choose one, and develop a rough plan for the board and the rules. A great variety of games resulted[4].

Stage 3: Develop and implement the plan in detail. Each group of students produces a detailed design, makes it, and checks the finished version to see it works well, revising if necessary.

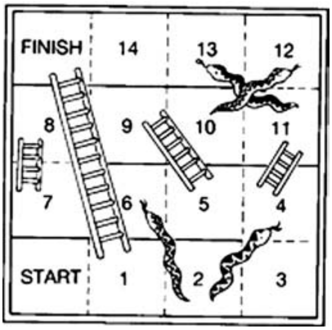
Stage 4: Each group evaluates the things that the other groups have produced. The groups exchange games and play them, and write comments. When they are returned, each group re-assesses its own game in the light of another group's comments. The class may or may not vote for favorites.

The design skill, experience and effort exemplified here were matched by an iterative sequence of trials in increasingly representative classrooms. This richness of feedback at all stages is the main difference between these research-based methods and the traditional approach, which relies on the extrapolation of craft-based skills to new situations. Extrapolation is generally unreliable, which is why most fields of product development (engineering, medicine,…) use research-based methods

Assessment in NTPS also follows an unusual model. Embedded assessment tasks test each student's understanding of the ongoing work—important for group projects. For example, the embedded assessment tasks in Stage 1 include the following game:

Snakes and Ladders. This is a game for two players. You will need a coin and two counters:

- Take turns to toss the coin. If it is heads, move your counter 2 places forward. If it is tails, move your counter 1 place forward.
- If you reach the foot of a ladder, you must go up it. If you reach the head of a snake, you must go down it.
- The winner is the first player to reach 'FINISH.'



External examinations, some months after the module work, assess their ability to transfer what they had learned to more or less similar problem contexts, within the same domain (here board games) or in structurally related areas. Two general points on assessment design are worth noting:

- The students' common experience of working on the module gives the assessment task designer some control over the transfer distance.
- Rich and open tasks allow responses at a wide range of levels; this is commonly used in other subjects (essays, for example) but underexploited in mathematics.

To summarize: each NTPS module provides learning and teaching materials, with embedded and external assessment; students work in small groups over three weeks per module; each module has real outcomes, with the class evaluating other groups' products. The other four modules are: *Be a Shrewd*

Chooser—how to make better consumer decisions; *Plan a Trip*—for the whole class out of school; *Produce a Game Show*—design and put on a TV quiz; *Be a Paper Engineer*—design pop-up cards and boxes. Students see this type of work as relevant to their current and future lives, especially when everyday contexts *(Shrewd Chooser* or *Trip*) are mixed with others that have an element of fantasy (*Quiz Show*, *Paper Engineer* and *Board Game*) but develop similar process skills.

Evaluation

The students viewed this as a serious enterprise, working together to develop a product they could be proud of. Most were motivated to take responsibility for the quality of their own and their group's work. Post interviews showed that nearly all students found the work interesting, challenging and enjoyable[5]. When asked to compare this work with "what you normally do in maths," their reaction was surprisingly strong; some groups burst out laughing, explaining that no-one could see their normal mathematics as anything other than a boring imposition. (Their teachers were *not* weak teachers of mathematics.)

Teacher reaction to NTPS was almost as positive. They enjoyed and valued the experience. They were relieved at the end of a module to get back to less taxing teaching, but looked forward to the next module in a few months time. In the outcome, though the modules were developed with students across the ability range, they were used more with low-achieving students—anything that works well with them is welcome, while there is pressure for high-achievers to stick to the standard track.

Parental concerns were addressed with carefully structured parent's meetings. Though they welcomed the "relevance," they had concerns about soft options. These disappeared when they tried problems from the modules and compared their efforts with student work.

Future prospects

After a two decade gap, the British are again taking an active interest in QL. The Bowland Trust and the Government are funding a set of 3-5 lesson "case studies" on a wide range of "real world" topics. We have been asked to develop two: *Reducing road accidents* and *How risky is life?* These are both challenging but we are having fun! The materials are due out in early 2008 in electronic and print forms. There have been related developments in Germany and Denmark.

For the US, we are working with some others on a proposal to develop QL units, probably for submission to NSF in due course.

I hope that I have said enough to show how detailed design considerations and careful development can be crucial in the success or failure of general principles. Good engineering of the tools and processes is important in increasing the probability of large-scale success in implementation, for QL as for any profound innovation.

Implications for teaching style

As the discussion of classroom learning confirms, QL cannot be taught with the standard EEE (*explanation–example–exercises*) approach. What extra skills do teachers need? The key elements include:

- *welcoming the world beyond mathematics*, in particular the world of students' lives and their imaginations, in the way that teachers of English and other subjects do;
- *handling discussion in the class in a non-directive but supportive way*, so that students feel responsible for deciding on the correctness of their and others' reasoning and do not expect either answers or confirmation from the teacher;
- *giving students time and confidence* to explore each problem thoroughly, offering help only when the student has tried, and exhausted, various approaches (rather than intervening at the first signs of difficulty);
- *providing strategic and tactical guidance*, rather than showing students who have difficulties how to do the problem, or dividing it into pieces;
- *finding supplementary questions* that build on each student's progress and lead them to go further;
- *helping students to assume responsibility for their own work,* to check their reasoning and their answers and, in discussion with other students, to evaluate the quality of their work.

These are profound changes, implying a change in the "classroom contract" (Brousseau, 1997) of mutual expectations between teacher and students—a change that needs to be made explicit by the teacher[6]. Professionals cannot easily change their well-grooved rituals of practice. The available research shows that this takes time and growing experience based on well-engineered support. The specific foci above will help to provide teachers with a solid start along the path of developing the variety of their mathematical questioning, which is so important in helping students of all abilities reach new levels of achievement.

Among teachers, there is too-often an unfortunate correlation between knowing more mathematics and having an inward-looking view of it[7]. This will make QL an unwelcome challenge for some high school teachers. For elementary and middle school teachers, the challenges of including real world problems are not as great. They have lived in a less specialized world. However, teachers respond to the success of their students, particularly those who find the subject difficult. Their students will flourish in QL.

All this is challenging at first, but teachers who acquire these skills seem to continue to use them; they do not revert to traditional styles. Well-engineered materials can provide enormous support to teachers and students, whether in modeling or in pure mathematical problem solving. Such materials are essential for most teachers in their first few years of such teaching, if they are to achieve success.

The core of the professional development needed is for teachers to gain the same kind of experience of real problem solving as their students will, using much the same materials, and to reflect on the teaching style changes it demands—the focus of the next section.

Who should teach QL?

In the excellent book, *Mathematics and Democracy* (Steen, 2002) I was astonished to see the view that QL should not be taught by mathematics teachers as part of the mathematics curriculum, but become a cross-curriculum responsibility. I disagree for the following reasons:

- Teaching QL well is mathematically demanding8, even for mathematics teachers; those less well-prepared could not cope.
- Utility is the reason why mathematics has such a large slice of curriculum time; in this era of unpredictable future challenges, utility requires QL.
- The cultural importance of mathematics is surely not greater than, say, music; how can this alone justify so much more curriculum time?
- As the experience of statistical education9 has shown, it is extremely difficult to establish cross-curricular teaching—even harder than introducing a new component into an established subject. If QL is not taught in mathematics, it will not happen.
- QL facilitates the learning of mathematics.

Implications for teacher education

Now, to come at last to the point of this conference, what does this mean for teacher education, both pre-service and in-service? I am no expert in this field, but I am a keen and experienced observer. I know that there are many schools of education around the country and the world for whom the teaching style elements enumerated above (with the probable exception of the first) are already major aims of their programs. For them the main challenge is to enlarge the problem set they build into their courses in the way I have outlined.

QL will prove challenging to many teacher educators in mathematics, who themselves may not use much of their mathematics in their lives outside the classroom[10]. How many of us do 'back of the envelope' estimations to check the assertions of advertisers or politicians? How many would, as a juror or a lawyer, have queried the argument presented by the expert witness in the crib-death problem cited above—elementary though the mathematics is?

Again the way forward is for the teacher educators to gain the same kind of experience of real problem solving as will their students, using much the same materials.

Teachers, like students, benefit from learning constructively—inferring general principles from their own experience of handling specific examples. Our and others' experience favors a *sandwich model*. The essence of this well-established and powerful approach is reflection among teachers, guided by an expert leader and/or well-engineered materials, interleaved with work with students in their classrooms—hence the name. The sequence is:

- *Launch.* Teachers together go through the learning activity in the role of students, then discuss the experience and how they will handle it in the classroom.
- *Teach.* In their own classrooms, the teachers take their students through the activity, collecting samples of student work and, later, making notes on the experience.
- *Reflect.* In the next professional development session, teachers share their experiences and their students' work, reflecting on the learning activity, student responses to it, how they might handle it differently, its wider implication for later lessons in the unit and for other teaching.

This model gives teachers a constructive learning experience, provided it is well-engineered so that the challenges and issues arise in a controlled way, digestible in form and pace, from specific substantial problems. Malcolm Swan (2006) explains the research basis of the sandwich model in the following terms:

> "Even in the face of contradictory evidence teachers hold tenaciously onto existing practices. In his literature review, Calderhead (1996) notes how pre-service teachers become more liberal and child-centered during training and then revert to control-oriented belief systems when they enter their full-time career. When well-grooved

practices are challenged, then teachers may react both affectively and cognitively. Any attempt to deconstruct someone's beliefs and practices through argument may be perceived as an attack on his or her own identity. Beliefs are more likely to be changed through reflecting on experience than through persuasion. It is only through making pre-existing experiences explicit, challenging them and offering opportunities to examine, elaborate, and integrate new experiences that teachers' behaviors are likely to change.

The situated nature of beliefs may thus mean that it is possible for teachers to adopt a new belief system in a restricted domain, or at least 'suspend disbelief' and act as if they believed differently. They may then subsequently reflect on the experience and accommodate or reject this new belief at least in a tentative way until it may be further tested.

This suggests that we cannot seek to change someone's beliefs so that they will behave differently. Rather, we encourage them to behave differently so that they may have cause to reflect on and modify their beliefs (Fullan, 1991, p.91). Teachers also need the support and resources to experience new ways of working. In the light of this, I suggest the following principles on which I based the development of an in-service program:

- Establish an informal candid culture in which existing beliefs are recognized, made explicit and are worked on in a reflective, non-judgmental atmosphere.
- Illustrate vivid, contrasting practices and discuss the beliefs that underpin these. These may provide 'challenge' or 'conflict.'
- Ask teachers to 'suspend' disbelief and act in new ways, 'as if they believed differently.' Offer mentor and a network of support as they do this.
- Encourage teachers to meet together and reflect on their new experiences and the implications that these offer.
- Ask teachers to reflect on and recognize the growth of new beliefs."

For in-service professional development the sandwich model is often straightforward to organize—participant teachers have their own classes and usually, with discretion and a reasonably supportive principal, can try new things with them.

In view of my inexperience in pre-service teacher preparation programs, my suggestions must be modest but introducing QL here is surely more challenging. The student teacher is a guest in the school, observing and/or substituting for the class teacher in an established program. For schools in which QL is already part of the implemented curriculum, the negotiations should be straightforward. It is made easier for QL than more didactic programs because of the less directive roles needed for teaching QL. Team teaching is an ideal entry mode. For other schools, the best hope may be to 'sell' QL to the principal as an obvious lacuna in mathematics curricula that is now coming onto the agenda. "Wouldn't you like to see what it looks like in the classroom, giving your school a flying start?" The contribution of QL to learning mathematics itself, discussed below, should always be kept in view.

Live in-service teacher education is expensive, but some is essential. It can be made much more effective by using good 'DIY' materials that support ongoing activity among teachers in a school between whatever live sessions may be available and affordable. Past experience shows that providers of live professional development welcome such support for continuing professional development. Is this true for teacher preparation programs?

Assessing teacher effectiveness raises a range of issues beyond the space I have here. I would like to make one obvious point—any methodology should look for changes in a teacher's classroom behavior and relate them to the list enumerated above. Well-designed structured observation before, during and after professional development is rarely part of the development process or of subsequent evaluation[11]; when it is used, it generally leads to radical redesign of the professional development along the lines sketched here.

QL and pure mathematics—not a zero-sum game

Anyone who argues for adding a new element to the curriculum must address the fact that it is seen as already overcrowded. Mathematics texts now have far more pages in them than any teacher can use[12]. One could therefore argue that to add more 'goodies' to the pile changes nothing, except marginally to decrease the already-small chance of anything new being used. More positively, *a significant amount of work on QL can actually reduce the overcrowding* by reducing the large amount of time (up to 35%) spent *re-teaching* concepts and skills (an ineffective approach to remedying misconceptions).

Modeling real world situations supports the learning of mathematical concepts and skills in several ways:

(a) It provides multiple *concrete embodiments* of mathematical concepts;

(b) It builds fluent *translation skills* between different representations;

(c) It involves *extended chains of reasoning*; for which

(d) It requires *procedural accuracy*, so encourages *checking* (not moving on regardless).

Indeed, many active proponents of modeling are mainly motivated by its power in teaching mathematics better. The Freudenthal Institute team, for example, say explicitly that the prime goal of their *Realistic Mathematics Education* (*RME*) approach is deeper understanding of mathematics itself. *Mathematics in Context* materials show this focus. The contributions of modeling to other mathematical competencies is surveyed in ICMI Study 14 (Blum, Galbraith, Henn, & Niss, 2007) in Section 3.4 which I (somewhat ironically, given my priorities) was asked to co-edit.

To illustrate (a), let us return to *The Great Horse Race* discussed earlier and the discussion by Swan et al (*ibid*. Section 3.4.1) of the concept formation it supports.

> All students quickly recognize that Horse 1 is not a good bet! A few have the misconception that, because there are two dice, higher numbered horses will move more rapidly. Even they, through playing the game, soon realize that horses in the middle will move faster "because there are more ways of making 7 than 11 or 12." Most will enumerate these "1+6, 2+5, 3+4..."; some initially miss that 4+3 is different from 3+4 until the teacher suggests two different colored dice. More advanced students consider the effect of the length of the track on the likely outcome. This game thus proves an effective (and quick) stimulus to concept formation. Furthermore if every student colors the squares traversed by each horse, they obtain a frequency distribution. When these are displayed around the classroom, students have an immediate visual of the variability of sample data. If the teacher enters these data in a spreadsheet, the frequency distribution of the totals will provide a better fit to the theoretical probabilities than will those of individual students.

The detailed design of this activity, guided by the student book, is crucial to its power and range. Its robustness in classroom use, not to mention some design ideas, comes from revisions, each guided by the feedback from successive rounds of closely observed trials. *In situated learning the quality of the situation is crucial.*

On translation skills, (b) above, when students work on modeling tasks, they express their thoughts in a variety of representations: words, diagrams, tables, spreadsheets, graphs, algebraic expressions. Indeed, many real world problems begin with information provided in a variety of representations. Since any representational medium will highlight some aspects of the structure, while obscuring others, students who engage in modeling activities develop *translation skills*—the ability to move information between different representations.

Items (c) and (d) above are self-evident. On (d), note how the pay-off from checking changes from short items to longer chains of reasoning which, if they are not correct throughout, leave you completely at sea.

Shorter modeling activities, too, have a part to play in both QL and the developing of robust mathematical skills. The interpretation of graphical information, reflecting (b), is one example (see Swan *et al*., 1986)

Finally, it is important to absorb the distinction, summarized in Figure 2, between:

- *standard applications* of a mathematical topic, and
- *active modeling* of a *non-routine* practical situation, to which several mathematical tools usually contribute.

Each is a necessary part of QL but they have different roles.

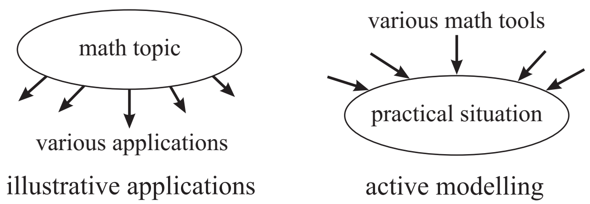


Figure 2. Standard illustrative applications v non-routine active modelling

Standard illustrative applications provide links between the real world problems and your mathematics toolkit. They can often be adapted to help with new problems—indeed all problem solving involves recognizing familiar structural features in a problem situation. However, tackling new problems effectively depends on experience with the processes of active modeling, supported by some teaching of the strategies and tactics that help in using one's mathematical toolkit to be more quantitatively literate.

A few key points

The following points may be worth restating:

- For QL, as with all situated learning, the quality of the situation is central to success.
- Most teachers will need at least as much support, in both materials and live teacher education, as they expect in familiar areas of mathematics teaching.

- High quality in both materials and processes depends on combining research awareness, imaginative design, and careful development of products and processes— an engineering research approach.
- Teacher education for this less familiar mathematical competency should be through constructive learning, involving teachers and teacher educators in the mathematical activities, linked to reflection on their implications for both the subject and its teaching.
- QL will help students learn more pure mathematics more effectively, building deeper understanding, richer connections and greater accuracy. This is particularly true for weaker students, narrowing "the gap."[13]

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Appendix A: Functional Mathematics for Educated Adults

A few thought-provoking tasks that any well-educated adult could, and should, be able to do without having been taught the specific problem, selected from the Mathematics Assessment Resource Service (MARS), Shell Centre for Mathematical Education, University of Nottingham. Commentary on the tasks and responses to them appears at the end of the appendix.

Sudden Infant Deaths = Murder?

In the general population, about 1 baby in 8,000 dies in an unexplained “crib death”. The cause or causes are at present unknown. Three babies in one family have died. The mother is on trial. An expert witness says:

> One crib death is a family tragedy; two is deeply suspicious; three is murder. The odds of even two deaths in one family are 64 million to 1.

Discuss the reasoning behind the expert witness' statement, noting any errors, and write an improved version to present to the jury.

Conference Budget

Your job is to plan a conference budget, using a computer spreadsheet. You have already made a start:

(i) Complete the entries for Wednesday in column D.

(ii) Calculate appropriate totals in column E.

(The spreadsheet was on a computer.)

A	B	C	D	E
	College charges	**Number**	**@ £ each**	**£**
Monday	Buffet Supper	30	17.00	**0**
	Single En-suite Accommodation	30	40.00	**0**
Tuesday	Breakfast	30	8.00	**0**
	Morning Coffee	30	1.90	**0**
	Luncheon	30	15.00	**0**
	Afternoon tea	30	1.90	**0**
	Dinner served	30	50.00	**0**
	Single En-suite Accommodation	30	40.00	**0**
	Plenary Room	30	15.77	**0**
	Breakout rooms	2	85.10	**0**
Wednesday	Breakfast	30		**0**
	Morning Coffee	30		**0**
	Luncheon	30		**0**
	Afternoon tea	30		**0**
	No Dinner	30		**0**
	Single En-suite Accommodation	30		**0**
	Plenary Room	30		**0**
	Breakout rooms	2		**0**
Thursday	Breakfast	30	8.00	**0**
	Morning Coffee	30	1.90	**0**
	Luncheon	30	15.00	**0**
	Afternoon tea	30	1.90	**0**
	Dinner	30	17.00	**0**
	Single En-suite Accommodation	30	40.00	**0**
	Plenary Room	30	15.77	**0**
	Breakout rooms	2	85.10	**0**
Friday	Breakfast	30	8.00	**0**
			Total charges	**0**
			VAT	**0**
			Total	**0**

Elementary School Teachers

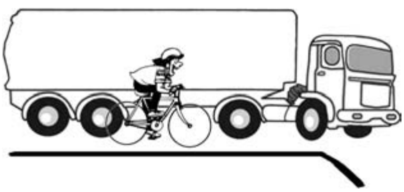
In a country with 300 million people, about how many elementary school teachers will be needed? Try to estimate a sensible answer using your own everyday knowledge about the world. Write an explanation of your answer, stating any assumptions you make.

Bike or Bus

Terry is soon to go to secondary school. There is no school bus. The bus trip costs $1 and Terry's parents are considering the alternative of buying him a bicycle.

Help Terry's parents decide what to do by carefully working out the relative merits of the two alternatives.

Right Turns



The truck is stopped at traffic lights, planning to turn right. The cycle is alongside.

If the cyclist waits for the truck to turn before moving, what will happen? Explain why this will happen with a diagram.

What would be your advice: to the truck driver? to the cyclist? Give reasons in each case.

Scheduling Traffic Lights

A new set of traffic lights has been installed at an intersection formed by the crossing of two roads. Right turns are *not* permitted at this intersection.

For how long should each road be shown the green light? Explain your reasoning clearly.

Being Realistic About Risk

"My sixty-year-old mother, who lives in New York, gets frightened by newspapers. One day she is afraid of being a victim of crime, the next she is frightened of being killed in a road accident, then it's terrorists, and so on."

(i) Use a website with national statistics to estimate the chances of my mother being a victim of the above events, and others you think she might worry about.

(ii) Write down some reassurance you would give her—and compare the likelihood of these events with the probability that women of her age will die during the coming year.

Commentary on the tasks, and responses to them:

Sudden Infant Deaths = Murder? What we expect here is not a full statistical analysis, which would need more information, but a recognition that the reasoning presented is deeply flawed. There are two elementary mistakes in the statement, and one that is a bit more subtle. It would be correct to say:

1. The chance of these deaths being entirely *unconnected* chance events is very small indeed—if there has been one death, the chance of two more unconnected deaths is about 64 million to one.
2. What can the connection be? It may be that the mother killed the children; on the other hand, particularly since we do not understand the cause(s) of crib death, there may be other explanations. For many conditions (cancer and heart disease, for example) genetic and environmental factors are known to affect the probability substantially.

Any lawyer or judge with functional mathematics should have seen problems with the witness statement. It is not lack of basic skills that was their failing (They could surely have worked out the chance of a double six on rolling two dice as 1/36) but an understanding of the necessary assumptions.

Conference Budget. This is a task we give (on a working spreadsheet) to candidates for the post of Secretary/Administrator in the team. Most are graduates. All "know Excel". None complete the task. Most see that Wednesday's values in Column D are probably the same as Tuesday's and Thursday's. Few enter the appropriate, or indeed any, formulas in Column E. (Formulating relationships is a basic piece of algebra that is neglected in schools—and mathematics tests.) Some even work out the row totals on a calculator, entering the *values*!

Elementary School Teachers. This kind of back-of-the-envelope calculation is an important life skill. Here it requires choosing appropriate facts (6 years in elementary school out of a life of 60-80 years, one teacher for 20-30 kids), and formulating appropriate proportional relationships giving (300*6)/(70*25) ~ 1 million primary teachers (to an accuracy appropriate to that of the data). This kind of linkage with the real world, common in the English language arts curriculum, is rare in school mathematics (and absent in tests).

Bike or Bus and Scheduling Traffic Lights. See *Ice Cream Van* in Appendix B.

Right Turns. Functional mathematics often involves space and shape, too.

Being Realistic About Risk. Education, and functional mathematics in particular, can help narrow the gap between perceived and real risk. Given the power of anecdote over evidence, exploited daily by the media, this is a major challenge; meeting it could make a huge contribution to people's quality of life, and that of their children. Few people have any sense of the magnitude of specific risks, or any idea of the unavoidable 'base risk' for someone of their age. (Note that only order-of-magnitude estimates, not accurate numbers, are relevant here.)

Explicitly teaching students to use their mathematics on real problems is now proven, with typical teachers; it is essential to functionality. These exemplars also show how deterministic and statistical reasoning intermesh in functional mathematics.

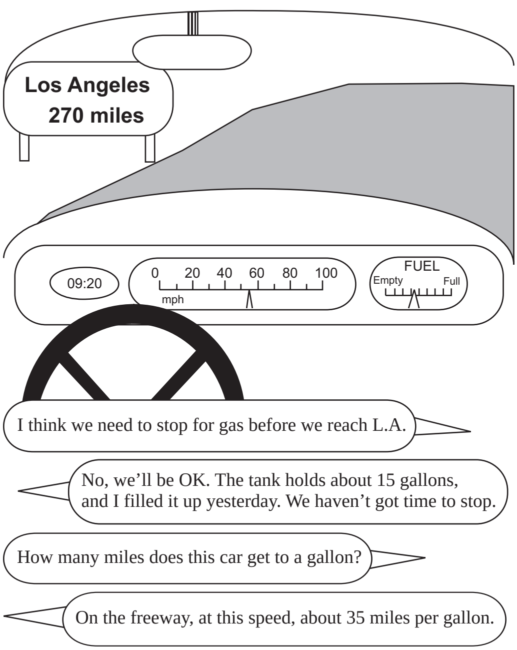
Appendix B
Functional Mathematics for Grade 8 or 9

A few thought-provoking tasks that any well-educated student should be able to do by age 15 without having been taught the specific problem, selected from the Mathematics Assessment Resource Service (MARS), Shell Centre for Mathematical Education, University of Nottingham. Commentary on the tasks and responses to them appears at the end of the appendix.

Freeway Journey

Referring to the figure below:

- Do they have to stop for gas? Explain your reasoning.
- Suppose they decide to stop for 10 minutes. At what time will they reach Los Angeles?



At the Airport

Den's Currency Exchange		
Currency	We Buy	We Sell
$ US Dollar	£ 0.533	£ 0.590
€ Euro	£ 0.660	£ 0.730
No commission!		

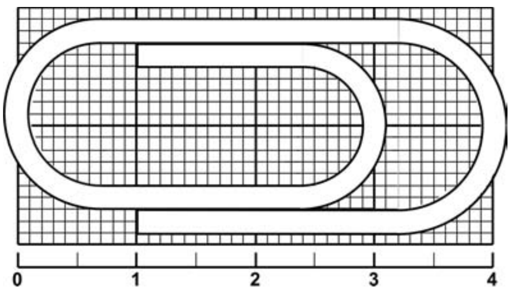
How many Euros (€) would you get for £500?

How many Pounds (£) can you get for $700?

How much would you have to pay, in Pounds and Pence, to get exactly €550?

Paper Clips

This paper clip is just over 4 cm long. How many paper clips like this may be made from a straight piece of wire 10 meters long?



Ice Cream Van

You are considering driving an ice cream van during the Summer break. Your friend, who "knows everything", says that "It's easy money". You make a few enquiries and find that the van costs $600 per week to hire. Typical selling data is that one can sell an average of 30 ice creams per hour, each costing 50c to make and each selling for $1.50.

How hard will you have to work in order to make this "easy money"? Explain your reasoning clearly.

Cold Calling

The following is part of a genuine letter of complaint to a bank.

> I would like to complain about the behavior of XYZ Bank and the advice given during a recent unsolicited telephone call. Having been told I was "pre-approved" for a $5,000 loan, the operator asked me for my financial details. I told her that I currently had two credit cards, one with a balance of $3000 and one with $1000. She said that they

could consolidate these debts into a single payment which would be cheaper. I pressed her on the APR which she explained was 16.4%, which caused me to decline the loan because my two credit cards are currently at 7% and 9.9% APR respectively. The operator then informed me that their loan would work out cheaper, because 7% and 9.9% works out at 16.9%, nearly 0.5% higher than the bank loan.

(i) Explain what is wrong with the operator's reasoning.

(ii) How much more expensive is the bank's consolidated loan?

Commentary on the tasks, and responses to them

Motorway Journey. From an actual test. Most examples of functional mathematics have been eliminated in the fragmentation of tasks to assess separate micro-skills.

At the Airport. It is interesting to compare this with a question from a current UK school test (on the right). Note how the simplification of the presentation leaves a major gap from real functionality. This unreality, characteristic of secondary school mathematics, confirms many students' view that the subject has no relevance to their lives.

The table shows the exchange rates between different currencies:

£1 (Pound)	is worth	€ 1.45 (Euros)
$1 (Dollar)	is worth	€ 0.81 (Euros)

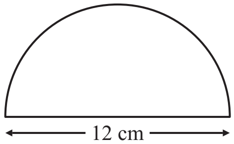
(a) Jane changes £400 into euros. How many euros does she receive?

(b) Sonia changes £672 euros into dollars. How many dollars does she receive?

Ice Cream Van. This task was used in a research study of the performance of 120 very able 17-year-old students. Many solved the tasks, using arithmetic and, sometimes graphs. *None used algebra*, the natural language for formulating such problems. Their algebra was non-functional, despite 5+ years of high success in the standard imitative inward-looking algebra curriculum.

Paper Clips. This task exemplifies a step towards functionality; a school mathematics version is shown on the right.

A semi-circle has a diameter of 12 cm. Calculate the perimeter.



Cold Calling. A common misconception, and con, to unravel. Explicitly teaching students to use their mathematics on real problems is now proven, with typical teachers; it is essential to functionality. These exemplars also show how deterministic and statistical reasoning intermesh in functional mathematics.

Endnotes

[1] Pure mathematical problem solving has a similar structure, though there are important differences.

[2] *Transfer distance* is a measure of how different two problems are, and so of how non-routine a task is, how far it differs from tasks with which the student is familiar. An important concept, no-one has seriously tackled the interesting challenge of inventing a practical way to quantify it, perhaps partly because it depends on the student's whole prior experience.

[3] I see the history of "problem solving" in US schools in the 1980s as an instance of this. After adoption by NCTM as a theme, much general advice was made available but little or no fully developed teaching material. Not much happened. At the Shell Centre we adopted a different approach, working with an examination board to develop coordinated *pressure* (new high-stakes exam tasks) and *support* (new teaching materials). These are published in (Shell Centre, 1984).

[4] In a few schools, the teachers decided to make this a joint project with the Art or Design departments, with excellent results. We encouraged this, but making this a requirement would have killed the project—and its effect on student attitudes to mathematics.

[5] The exceptions were a small proportion of normally high-achieving students who found being faced with a new 'game' of a different kind somewhat threatening.

[6] When we began to develop support for problem solving in pure mathematics (Shell Centre 1984), the first exploratory set of examples we gave to students was headed "THIS IS NOT A MATHS EXAM." It was, of course, but not what they expected of one.

[7] The evaluation of the USMES project (1969) found that mathematics teachers were the worst USMES teachers; the best were "drop-out Art teachers!" A case can be made for "style specialization" —teaching investigation is as far from traditional EEE mathematics teaching as teaching some other subjects. Let those math teachers who can do it, concentrate on it.

[8] "….the sophisticated use of elementary mathematics," in Lynn Steen's immortal phrase, is not something to expect from those with weak mathematics.

[9] Statistics educators have always seen QL as central and, mainly for this reason, sought to separate themselves from mathematics education; however, this separation is unhelpful. Many problems should be tackled deterministically, at least initially; sometimes the analysis must take random variation into account. (Statistics has no monopoly on data.)

[10] The "test for well-educated adults" in Appendix A is a useful self-evaluation tool.

[11] Indeed there are those who would regard classroom observation as unpardonably intrusive, regarding professional development as a civilized exchange between fellow professionals in which the sole criterion of success is whether the teacher found the experience valuable.

[12] This is primarily because they are designed, not for teachers, but for the state adoption processes of Texas and California. To be considered, let alone adopted, text packages have to check every box on a list that seems to be the union of the wishes of the members of the adoption committee. From this wish-list, teachers select what they want to use—often just what they know well.

[13] … but not guaranteeing the early fluency in *abstract* algebra that seems to be a prime current goal in the U.S.