Problems with Patterns and Numbers

Masters for Photocopying

Joint Matriculation Board
Shell Centre for Mathematical Education

Teaching Strategic Skills – Publications List

Problems with Patterns and Numbers – the "blue box" materials

- School Pack – *Problems with Patterns and Numbers* 165 page teachers' book and a pack of 60 photocopying masters.
- Software Pack – Teaching software and accompanying teaching notes. The disc includes SNOOK, PIRATES, the SMILE programs CIRCLE, ROSE and TADPOLES, and four new programs* KAYLES, SWAP, LASER and FIRST. Available for BBC B & 128, Nimbus, Archimedes and Apple II (* the Apple disc only includes the five original programs).
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- Software Pack – Teaching software and accompanying teaching notes. The disc includes TRAFFIC, BOTTLES, SUNFLOWER and BRIDGES. Available for BBC B & 128, Nimbus, Archimedes and PC.
- Video Pack – A VHS videotape with notes.

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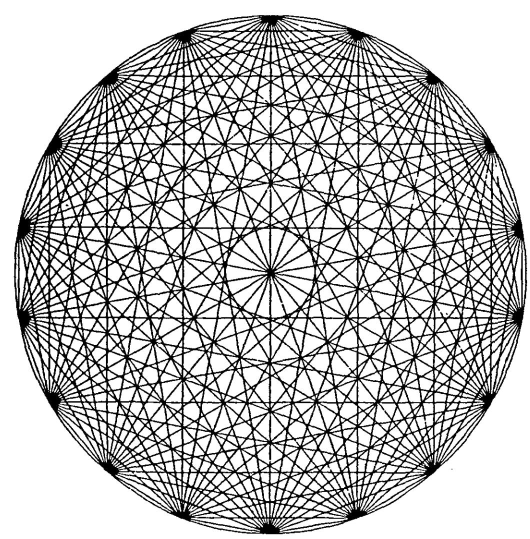
* The numbers in brackets refer to the corresponding pages in the Module book.

† The masters for A1, A2 and A3 should be used to form four page booklets, by photocopying back to back onto paper and folding this paper in half.

Specimen Examination Questions



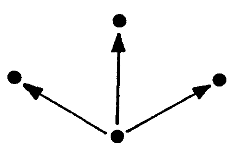
THE CLIMBING GAME

This game is for two players.

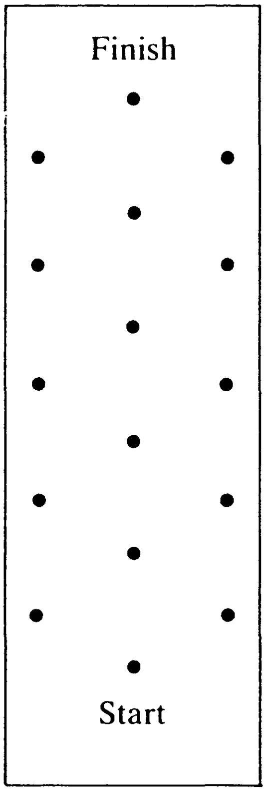
A counter is placed on the dot labelled "start" and the players take it in turns to slide this counter up the dotted grid according to the following rules:

At each turn, the counter can only be moved to an *adjacent* dot *higher* than its current position.

Each movement can therefore only take place in one of three directions:

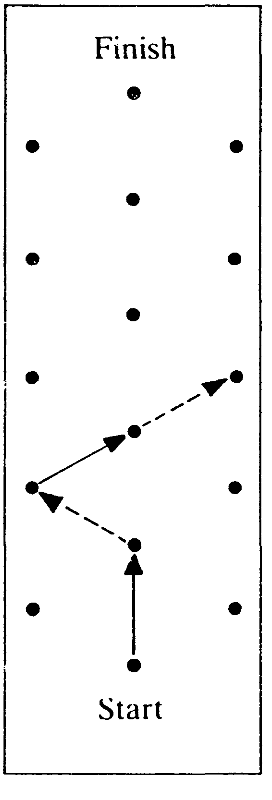


The first player to slide the counter to the point labelled "finish" wins the game.

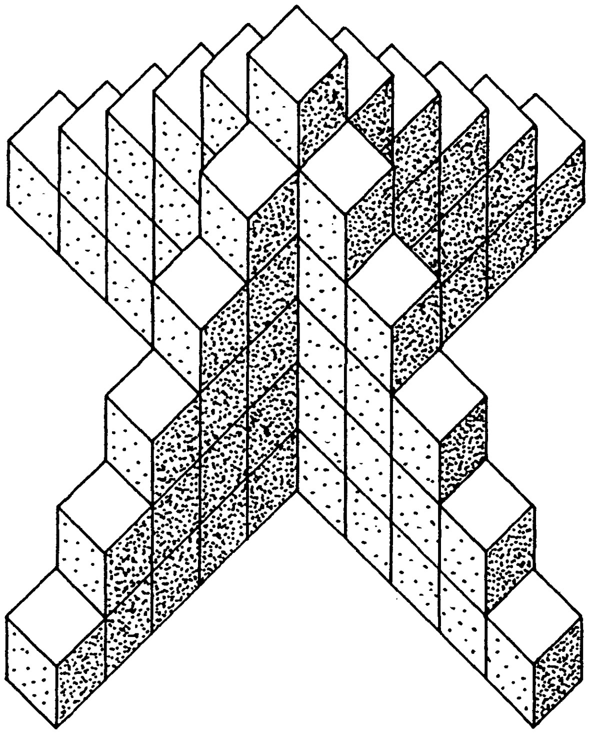


(i) This diagram shows the start of one game, played between Sarah and Paul.
Sarah's moves are indicated by solid arrows (———►)
Paul's moves are indicated by dotted arrows (– – -►)
It is Sarah's turn. She has two possible moves.
Show that from one of these moves Sarah can ensure that she wins, but from the other Paul can ensure that he wins.

(ii) If the game is played from the beginning and Sarah has the first move, then she can always win the game if she plays correctly.
Explain how Sarah should play in order to be sure of winning.



SKELETON TOWER



(i) How many cubes are needed to build this tower?

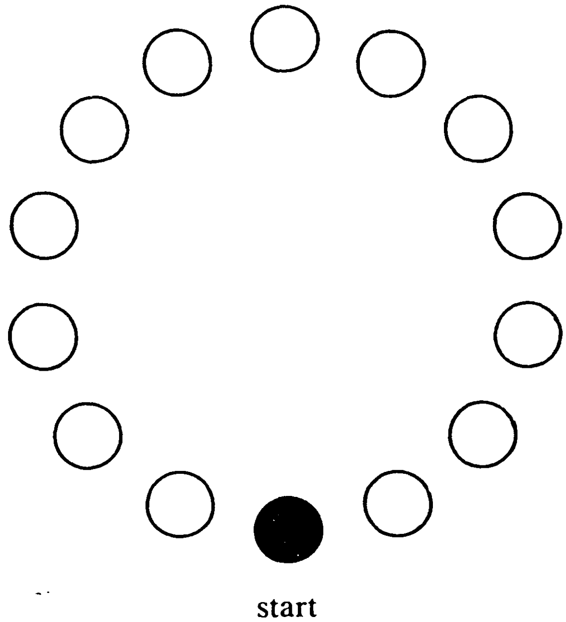
(ii) How many cubes are needed to build a tower like this, but 12 cubes high?

(iii) Explain how you worked out your answer to part (ii).

(iv) How would you calculate the number of cubes needed for a tower *n* cubes high?

STEPPING STONES

A ring of "stepping stones" has 14 stones in it, as shown in the diagram.



A girl hops round the ring, stopping to change feet every time she has made 3 hops. She notices that when she has been round the ring three times, she has stopped to change feet on each one of the 14 stones.

(i) The girl now hops round the ring, stopping to change feet every time she has made 4 hops. Explain why in this case she will not stop on each one of the 14 stones no matter how long she continues hopping round the ring.

(ii) The girl stops to change feet every time she has made n hops. For which values of n will she stop on each one of the 14 stones to change feet?

(iii) Find a general rule for the values of n when the ring contains more (or less) than 14 stones.

FACTORS

The number 12 has six factors: 1, 2, 3, 4, 6 and 12.
Four of these are even (2, 4, 6 and 12)
and two are odd (1 and 3).

(i) Find some numbers which have all their factors, except 1, even.
Describe the sequence of numbers that has this property.

(ii) Find some numbers which have exactly half their factors even. Again describe the sequence of numbers that has this property.

Explain in both part (i) and part (ii) why your result is true.

REVERSES

Here is a row of numbers: 2, 5, 1, 4, 3.

They are to be put in ascending order by a sequence of moves which reverse chosen blocks of numbers, *always starting at the beginning of the row.*

Example:

2, 5, 1, 4, 3	reversing the first 4 numbers gives	4, 1, 5, 2, 3
4, 1, 5, 2, 3	reversing the first 3 numbers gives	5, 1, 4, 2, 3
5, 1, 4, 2, 3	reversing all 5 numbers gives	3, 2, 4, 1, 5

. . . . .

. . . . .

. . . . .

. . . . .

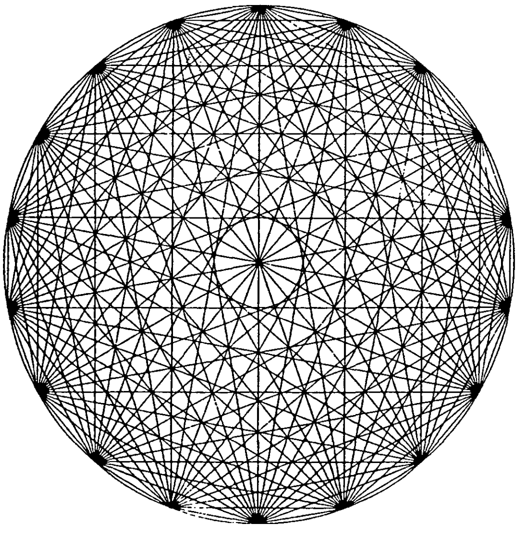
1, 2, 3, 4, 5

(i) Find a sequence of moves to put the following rows of numbers in ascending order

(a) 2, 3, 1
(b) 4, 2, 3, 1
(c) 7, 2, 6, 5, 4, 3, 1

(ii) Find some rules for the moves which will put any row of numbers in ascending order.

Classroom Materials



INTRODUCTORY PROBLEMS

These are different kinds of problem to those you are probably used to. They *do not* have just one right answer and there are many useful ways to tackle each of them. Your teacher is interested in seeing how well you can tackle these problems *on your own*. The methods you use are as important as the answers you get, so please *write down everything* you do, even if you are not sure it is right.

1 Target

On a calculator you are **only** allowed to use the keys

[3] [4] [×] [−] [=]

You can press them as often as you like.
You are asked to find a sequence of key presses that produce a given number in the display. For example, 6 can be produced by

$$3\times4-3-3=$$

(a) Find a way of producing each of the numbers from 1 to 10. You must "clear" your calculator before each new sequence.
(b) Find a second way of producing the number 10. Give reasons why one way might be preferred to the other.

2 Discs

(7) (10)

Here are two circular cardboard discs. A number is written on the top of each disc. There is another number written on the reverse side of each disc.
By tossing the two discs in the air and then adding together the numbers which land uppermost, I can produce any one of the following four totals:

11, 12, 16, 17.

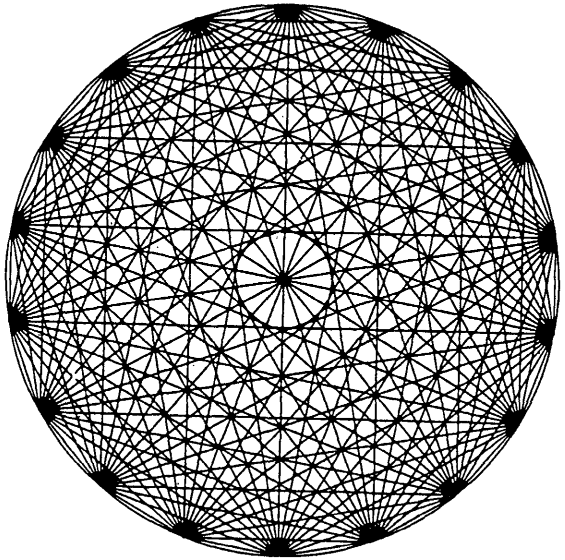
(a) Work out what numbers are written on the reverse side of each disc.

(b) Try to find a different solution to this problem.

3 Leagues

A top division has 22 teams. Each team plays all the other teams twice—once at home, and once away. Games are usually played on Saturdays, but sometimes on Wednesdays too. The season lasts about 35 weeks.
There is a proposal to expand this top division to 30 teams.
How many matches in all would be played, and how many matches would each team play? What would the effect be on the length of the playing season?

Mystic Rose



This diagram has been made by connecting all 18 points on the circle to each other with straight lines. Every point is connected to every other point . . .

How many straight lines are there?

Money

Suppose you have the following 7 coins in your pocket . . .

1p, 2p, 5p, 10p, 20p, 50p, £1

How many different sums of money can you make?

A1 ORGANISING PROBLEMS

The Tournament

A tournament is being arranged. 22 teams have entered. The competition will be on a league basis, where every team will play all the other teams twice—once at home and once away. The organiser wants to know how many matches will be involved.

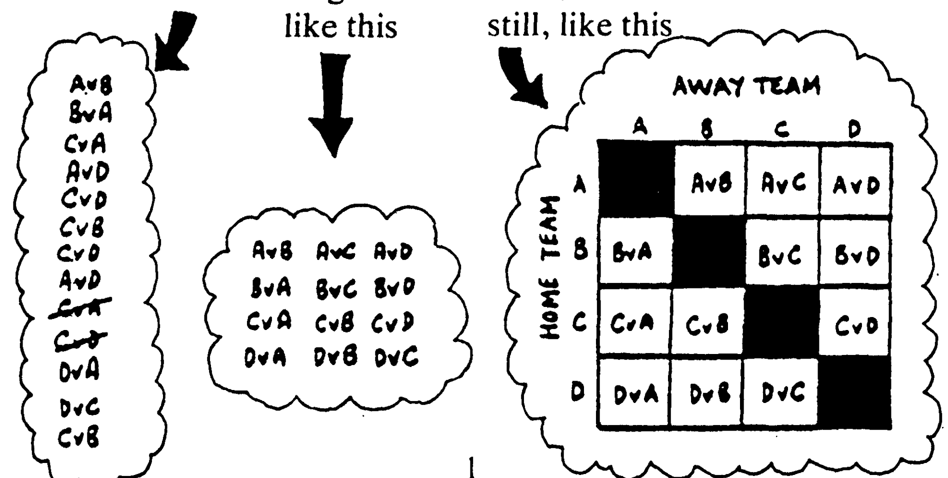
Often problems like this are too hard to solve immediately. If you get stuck with a problem, it often helps if you first **try some simple cases.**

So, suppose we have only 4 teams instead of 22.

Next, if you can **find a helpful diagram,** (table, chart or similar), it will help you to **organise the information systematically.**

For example,

Avoid this — Be organised like this — or, better still, like this



By now, you should be able to see that our 4 teams require 12 matches.

* How many matches will 6 teams require?
  How many matches will 7 teams require?
  Invent and do more questions like these.

* **Make a table** of your results. This is another key strategy . . .

Number of Teams	4	6	7			
Number of Matches	12					

* Try to **spot patterns** in your table.
  Write down what they are.
  (If you can't do this, check through your working, reorganise your information, or produce more examples.)

* Now try to **use your patterns** to solve the original problem with 22 teams.

* Try to **find a general rule** which tells you the number of matches needed for *any* number of teams. Write down your rule in words and, if you can, by a formula.

* **Check** that your rule always works.

* **Explain** why your rule works.

Now use the key strategies . . .

Try some simple cases

Find a helpful diagram

Organise systematically

Make a table

Spot patterns

Use the patterns

Find a general rule

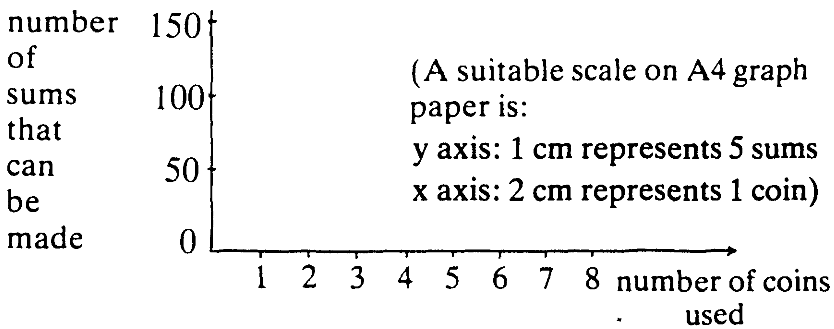
Explain why it works

Check regularly

to solve the problems on the next page . . .

Method 4 "Drawing a Graph"

* Draw a graph to show the relationship between the number of coins and the number of sums that can be made.



(A suitable scale on A4 graph paper is:
y axis: 1 cm represents 5 sums
x axis: 2 cm represents 1 coin)

* Try to use your graph to solve the problem for 7 coins.

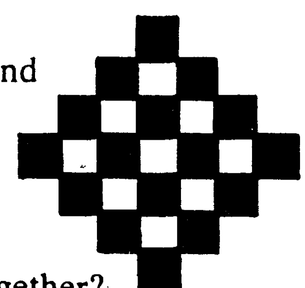
* **What are the advantages and disadvantages of these 4 methods?**

* **Can you invent any other methods?**

Try to solve the following problem using four different methods.
Which method do you prefer for this problem? Why?

Town Hall Tiles

This pattern is made up of black and white tiles. It is 7 tiles across.



In the Town Hall there is a pattern like this which is 149 tiles across.
How many tiles will it contain altogether?

A2 TRYING DIFFERENT APPROACHES

In the last lesson, you looked at the following problem . . .

Money

Suppose you have the following 7 coins in your pocket . . .

1p, 2p, 5p, 10p, 20p, 50p, £1.

How many different sums of money can you make?

We will now look at different ways of solving this problem and compare their advantages and disadvantages.

Method 1 "Method of 'differences'"

* Continue the table below for a few more terms:

Number of coins used	1	2	3	
Number of sums that can be made	1	3	7	

Differences: 2 4

* Explain where the numbers 1, 3 and 7 in the table come from.
(Do these numbers depend on the particular coins that are chosen?)

* Try to see a pattern in the differences between successive numbers in the table.
How will this pattern continue?
Are you sure? Try to explain it.

* Solve the problem with the 7 coins using this method.

Method 2 "Systematic Counting"

							sum made
						①	1p
					②		2p
					②	①	3p
				⑤			5p
				⑤		①	6p
				⑤	②		7p
				⑤	②	①	8p
			⑩				10p
			⑩			①	11p
			⑩		②		12p
			⑩		②	①	13p
			⑩	⑤			15p
			⑩	⑤		①	16p
			⑩	⑤	②		17p
			⑩	⑤	②	①	18p
		⑳					20p
		⑳				①	21p
(£1)	(50)	⑳	⑩	⑤		①	£1·86
(£1)	(50)	⑳	⑩	⑤	②		£1·87
(£1)	(50)	⑳	⑩	⑤	②	①	£1·88

2

This diagram shows a systematic attempt to list all the possible sums of money that can be made. (There is insufficient room to reproduce it all!)

* Try to see a quick way of counting the number of different sums.
* Solve the problem with 7 coins using this method.

Method 3 "Finding a Rule"

* Try to find a rule which links the number of coins with the number of sums of money directly.

Number of coins used	Rule ?	Number of sums that can be made
1	? →	1
2	? →	3
3	? →	7
4	? →	. . .
7	? →	. . .

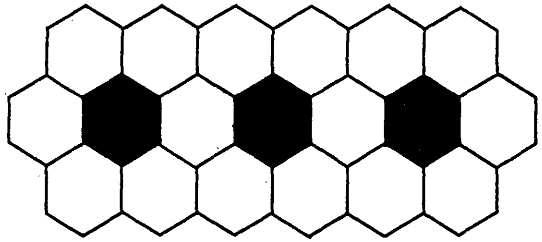
* Try to express your rule in words. Check that your rule always works.
  If you can, express your rule as a formula.
* Solve the problem for 7 coins using your rule.

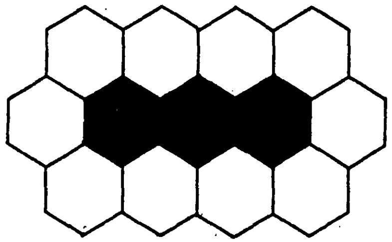
3

INVENTING A PROBLEM

Flower Beds

There are many other ways of surrounding flower beds with hexagonal paving slabs.
Invent your own examples and try to find formulae.





House of Cards

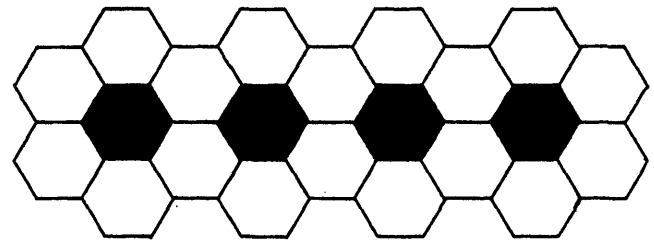
Try to think of other ways of constructing houses of cards. Draw them.

How many cards would you need in order to break the world record using your system?

A3 SOLVING A WHOLE PROBLEM

Try to solve the following problem using all you have learnt.
A list of strategic hints is provided over the page.

Flower Beds



The council wish to create 100 flower beds and surround them with hexagonal paving slabs according to the pattern shown above. (In this pattern 18 slabs surround 4 flower beds.) How many slabs will the council need? Find a formula that the council can use to decide the number of slabs needed for any number of flower beds.

* **Try some simple cases.**

* **Organise them systematically** using a **helpful diagram** or other representation.

* Try other simple cases and **make a table** of your results.

* **Look for patterns** in your table. Write them down in words.

* **Use these patterns to find a general rule.** Try to write your rule down in words. If you can, express your rule as an algebraic formula. **Check that your rule always works.**

* Try to **explain** why your rule works.

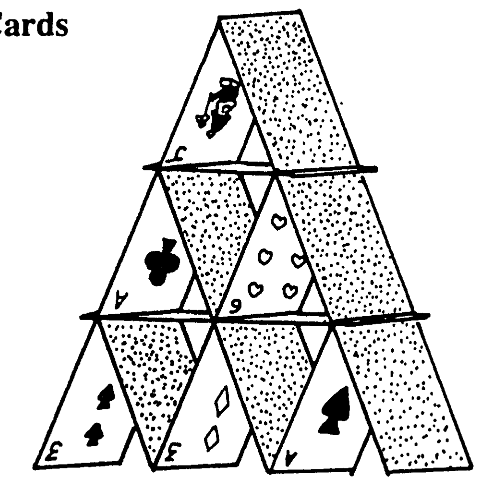
* Use your rule to find out how many paving slabs will be needed for 100 flower beds.

* Use your rule to write down the number of paving slabs needed for *n* flower beds.
(There are several ways of doing this. Try to find some alternatives.)

* If you get stuck—**try a different approach.**

Now try this problem in a similar way:

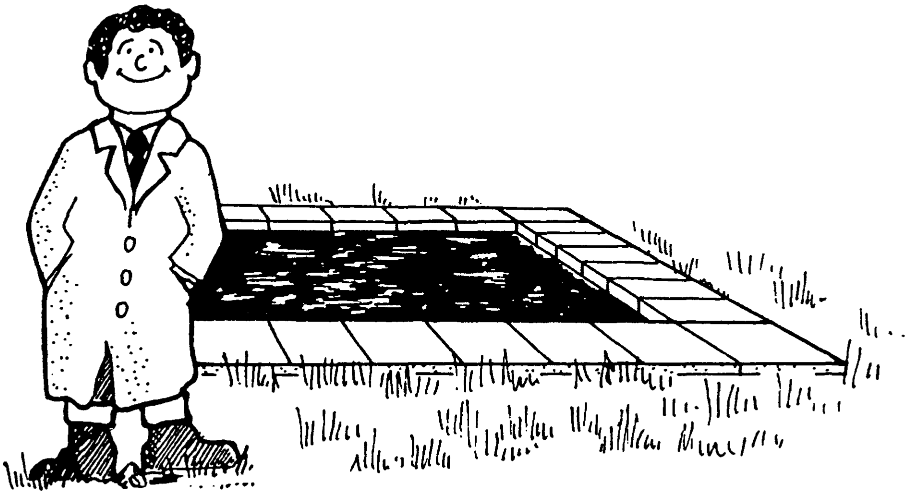
House of Cards



This house of cards is 3 storeys high.
15 cards are needed.

* How many cards would be needed for a similar house, 10 storeys high?
* The world record for the greatest number of storeys is 61. How many cards would you need to break this record, and make a house 62 storeys high?

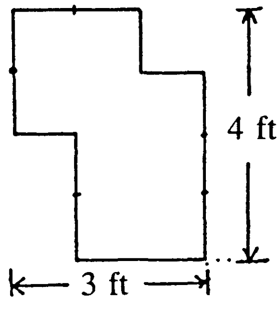
POND BORDERS



Joe works in a garden centre that sells square ponds and paving slabs to surround them. The paving slabs used are all 1 foot square.

The customers tell Joe the dimensions of the pond, and Joe has to work out how many paving slabs they need.

* How many slabs are needed in order to surround a pond 115 feet by 115 feet?
* Find a rule that Joe can use to work out the correct number of slabs for any square pond.
* Suppose the garden centre now decides to stock rectangular ponds. Try to find a rule now.
* Some customers want Joe to supply slabs to surround irregular ponds like the one below:—



(This pond needs 18 slabs. Check that you agree).

Try to find a rule for finding the number of slabs needed when you are given the overall dimensions (in this case 3 feet by 4 feet).

Explain why your rule works.

POND BORDERS . . . PUPIL'S CHECKLIST

Try some simple cases	* Try finding the number of slabs needed for some small ponds.
Be systematic	* Don't just try ponds at random!
Make a table	* This should show the number of slabs needed for different ponds. (It may need to be a two-way table for rectangular and irregular ponds).
Spot patterns	* Write down any patterns you find in your table. (Can you explain *why* they occur?) * Use these patterns to extend the table. * Check that you were right.
Find a rule	* Either use your patterns, or look at a picture of the situation to find a rule that applies to any size pond.
Check your rule	* Test your rule on small and large ponds. * Explain *why* your rule always works.

THE "FIRST TO 100" GAME

This is a game for two players.
Players take turns to choose any whole number from 1 to 10.
They keep a running total of all the chosen numbers.
The first player to make this total reach exactly 100 wins.

Sample Game:

Player 1's choice	Player 2's choice	Running Total
10		10
	5	15
8		23
	8	31
2		33
	9	42
9		51
	9	60
8		68
	9	77
9		86
	10	96
4		100

So Player 1 wins!

Play the game a few times with your neighbour.
Can you find a winning strategy?

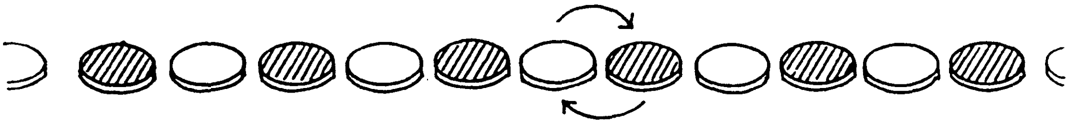
* Try to modify the game in some way, e.g.:
- — suppose the first to 100 *loses* and overshooting is not allowed.
- — suppose you can only choose a number between 5 and 10.

THE "FIRST TO 100" GAME . . . PUPIL'S CHECKLIST

Try some simple cases	*	Simplify the game in some way: e.g.:— play "First to 20" e.g.:— choose numbers from 1 to 5 e.g.:— just play the end of a game.
Be systematic	*	Don't just play randomly!
	*	Are there good or bad choices? Why?
Spot patterns	*	Are there *any* positions from which you can always win?
	*	Are there other positions from which you can always reach these winning positions?
Find a rule	*	Write down a description of "how to always win this game". Explain why you are sure it works.
	*	Extend your rule so that it applies to the "First to 100" version.
Check your rule	*	Try to beat somebody who is playing according to your rule.
	*	Can you convince *them* that it always works?
Change the game in some way	*	Can you adapt your rule for playing a new game where: — the first to 100 *loses*, (overshooting is not allowed) — you can only choose numbers between 5 and 10. — . . .

SORTING

50 red and 50 blue counters are placed alternately in a line across the floor: RBRBRBR . . . RB



By swapping adjacent counters (see arrows) they have to be sorted into 2 groups, with all the reds at one end and all the blues at the other: RRR . . . RRRBBB . . . BBB

* What is the *least* number of moves needed to do this?
How many moves are needed for n red and n blue counters?

* What happens when the counters are placed in different starting formations:
For example RRBBRRBBRRBB . . . RRBB
or RBBRRBBRRBB . . . RBBR

* What happens when there are red, blue and green counters arranged RBGRBG . . . RBG
What happens with 4 colours?
What happens with m colours?

* Invent and explore your own arrangement of counters.
Write about your findings.

SORTING . . . PUPIL'S CHECKLIST

Try some simple cases	*	Try finding the number of moves needed for just a few counters.
Be systematic	*	Try swapping counters systematically.
Find a helpful representation	*	If you are unable to use real counters, can you find a simple substitute?
	*	Can you use the simple cases you have already solved, to generate further cases by adding extra pairs of counters rather than starting from the beginning each time?
Make a table	*	Make a table to show the relationship between the number of counters and the number of swaps needed.
Spot patterns	*	Write about any patterns you find in your table. (Can you explain why they occur?)
	*	Use these patterns to extend the table.
	*	Check that you were right.
Find a rule	*	Use your patterns, or your representation, to find a rule that applies to any number of counters.
Check your rule	*	Test your rule on small and large numbers of counters.
	*	Try to explain *why* your rule must always work.

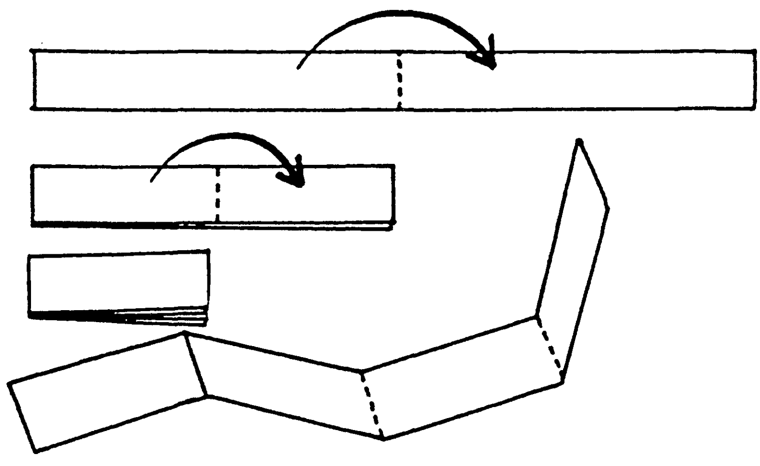
PAPER FOLDING

For this investigation, you will need a scrap of paper.
Fold it in half, and then in half again. In both cases you should fold left over right. Open it out and look at the folded creases:

first fold

second fold

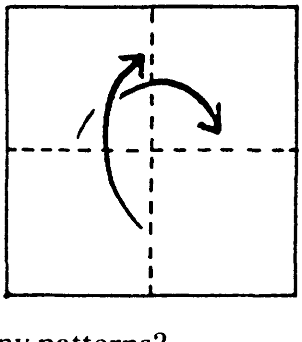
now unfold:



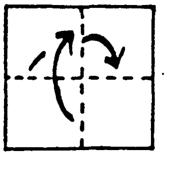
You should see 3 creases — one "up" and two "down".

* Now suppose you were able to fold your paper strip in half, left over right, 6 times, and then unfold it completely.
  Predict the total number of creases you would get.
  How many of these are "up" creases and how many are "down"?
  What order would these creases come in?
* Explain how you can predict the number and order of creases for 7, 8, . . . folds.
* Try folding the paper in a different way and explore the patterns in the positioning and number of your creases. Write about your findings.
  For example, here is a tricky two-step case . . .

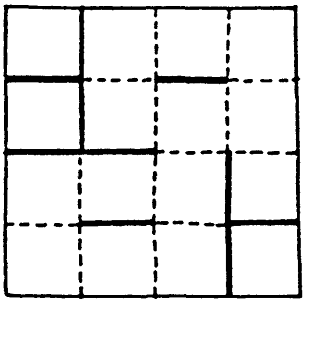
Left to right *then*
Bottom to top . . .



and
again . . .



and
unfold . . . (gasp!)



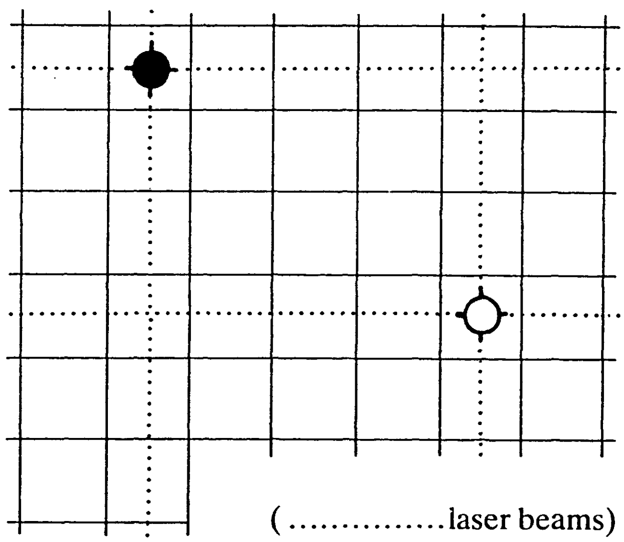
Any patterns?

PAPER FOLDING . . . PUPIL'S CHECKLIST

Try some simple cases	* It is very difficult to fold a normal sheet of paper in half 6 times. (Just think how thick it will be!), so try just a few folds first.
Be systematic	* Make sure that you always fold from left to right — don't turn your paper over in between folds!
Find a helpful representation	* Invent symbols for "up" and "down" creases. * Use your symbols to record your results.
Make a table	* Make a table to show the relationship between the number of times the paper is folded and the number of upward and downward creases, and also the *order* in which these creases occur.
Spot patterns	* Write about any patterns you find in your table. Can you explain *why* they occur? * Use these patterns to extend the table. * Check that you were right.
Find a rule	* Use your patterns to find rules that apply to any number of folds.
Check your rule	* Test your rules on large and small numbers of creases. * Try to explain *why* they work.
Extend the problem	* Invent your own system of folding. * Try to *predict* what will happen, then check to see if you were right.

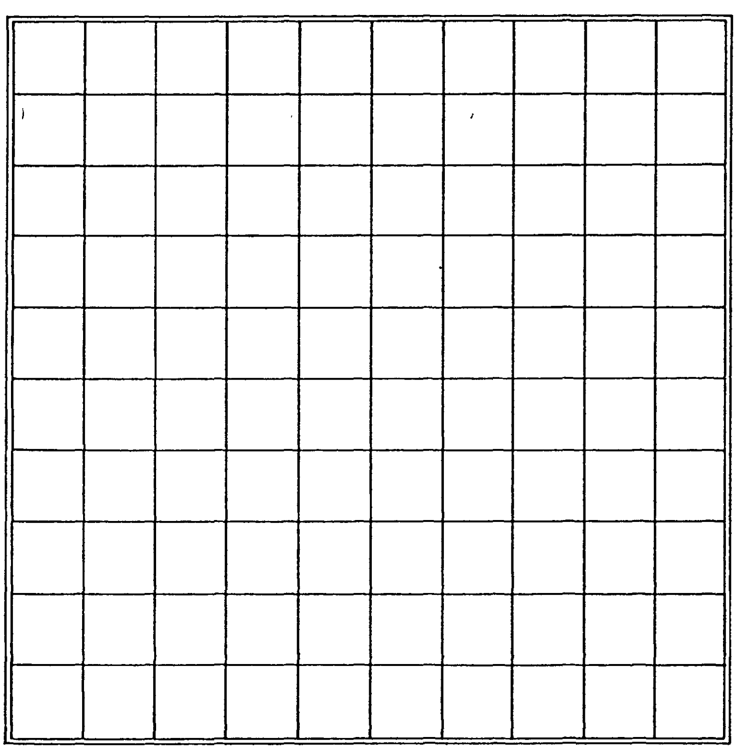
LASER-WARS

 and  represent two tanks armed with laser beams that annihilate anything which lies to the North, South, East or West of them. They move alternately. At each move a tank can move any distance North, South, East or West but cannot move across or into the path of the opponent's laser beam. A player loses when he is unable to move on his turn.



(..............laser beams)

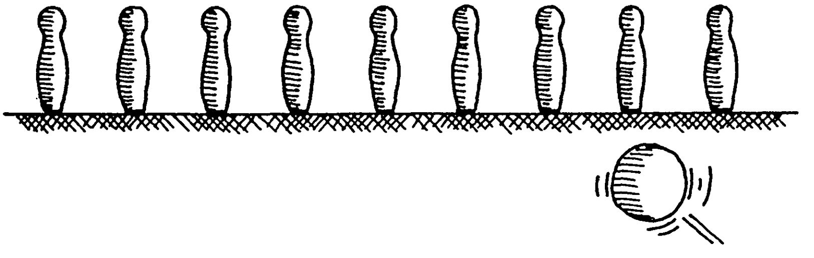
* Play the game on the board below, using two objects to represent the tanks. Try to find a winning strategy, which works *wherever* the tanks are placed to start with.



* Now try to change the game in some way . . .

KAYLES

This is like an old 14th century game for 2 players, in which a ball is thrown at a number of wooden pins standing side by side:



The size of the ball is such that it can knock down either a single pin or two pins standing next to each other. Players alternately roll a ball and the person who knocks over the last pin (or pair of pins) wins.

Try to find a winning strategy. (Assume that you can always hit the pin or pins that you aim for, and that no one is ever allowed to miss).

Now try changing the rules . . .

CONSECUTIVE SUMS

The number 15 can be written as the sum of consecutive whole numbers in three different ways:

$$15=7+8$$
$$15=1+2+3+4+5$$
$$15=4+5+6$$

The number 9 can be written as the sum of consecutive whole numbers in two ways:

$$9=2+3+4$$
$$9=4+5$$

Look at numbers *other than* 9 and 15 and find out all you can about writing them as sums of consecutive whole numbers.

Some questions you may decide to explore . . .

Which numbers *cannot* be written as consecutive sums?

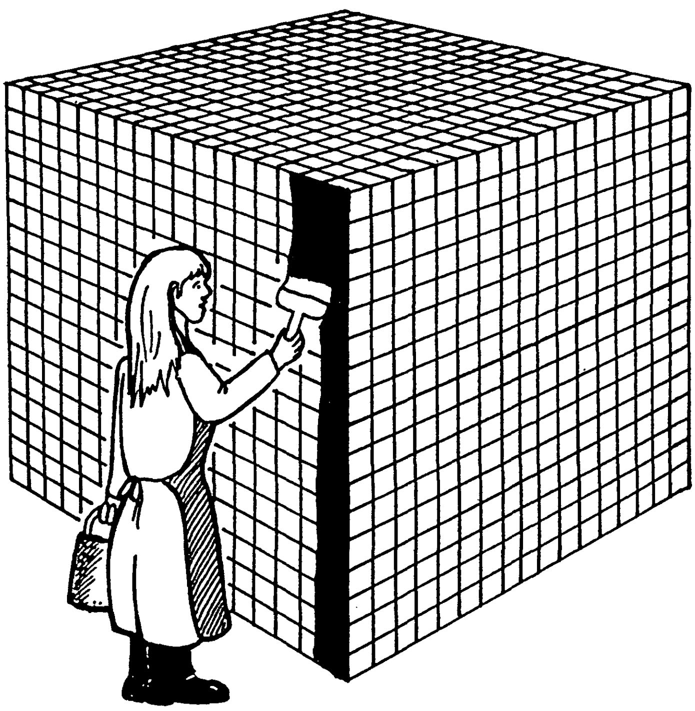
What kinds of numbers can be written as the sum of 2 or 3 or 4 or . . . consecutive numbers?

How many ways can various numbers be produced?

Spaces for your own questions when you think of any.

Write about your discoveries.
Try to explain *why* they occur.

THE PAINTED CUBE



* Imagine that the six outside surfaces of a large cube are painted black. This large cube is then cut up into 4,913 small cubes. ($4,913=17\times17\times17$).

How many of the small cubes have:

0 black faces?
1 black face?
2 black faces?
3 black faces?
4 black faces?
5 black faces?
6 black faces?

* Now suppose that you cut the cube into n^3 small cubes . . .

SCORE DRAWS

"At the final whistle, the score was 2—2"

What was the half time score? Well, there are nine possibilities:

0—0; 1—0; 0—1; 2—0; 1—1; 2—1; 2—2; 1—2; 0—2

* Now explore the relationship between other drawn matches, and the number of possible half-time scores.

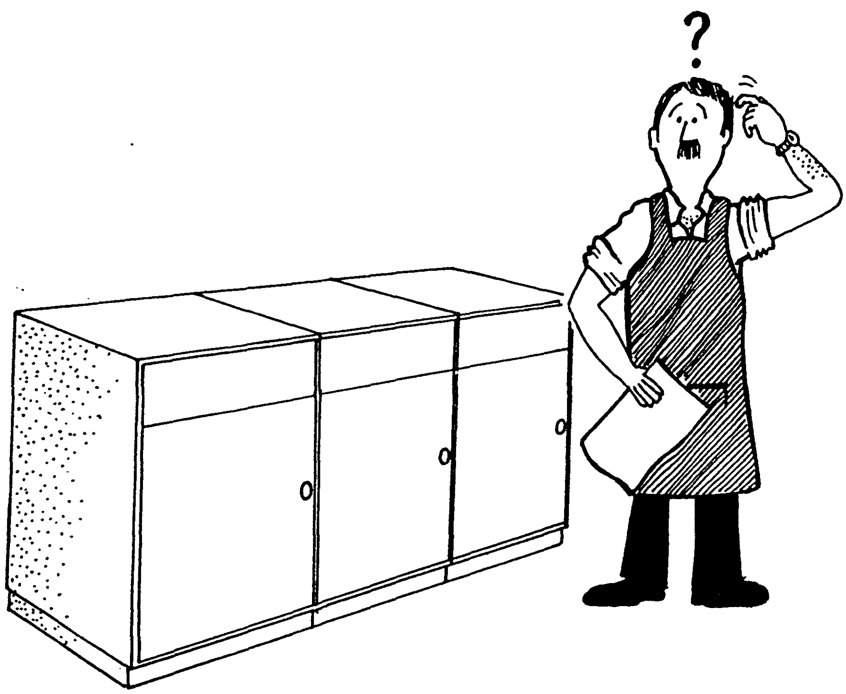
There are six possible ways of reaching a final score of 2—2:

1.	0—0,	1—0,	2—0,	2—1,	2—2
2.	0—0,	1—0,	1—1,	2—1,	2—2
3.	0—0,	1—0,	1—1,	1—2,	2—2
4.	0—0,	0—1,	1—1,	2—1,	2—2
5.	0—0,	0—1,	1—1,	1—2,	2—2
6.	0—0,	0—1,	0—2,	1—2,	2—2

* How many possible ways are there of reaching other drawn matches?

* Finally, consider what happens when the final score is *not* a draw.

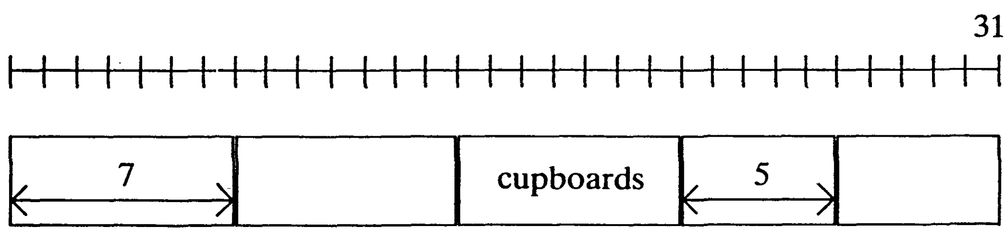
CUPBOARDS



A factory sells cupboards in two standard widths: 5 dm and 7 dm.
(Note: 1 dm=1 decimetre=10 centimetres).

By placing combinations of these cupboards end to end, they can be fitted into rooms of various sizes.

For example, two 5 dm and three 7 dm cupboards can be fitted into a room 31 dm long.



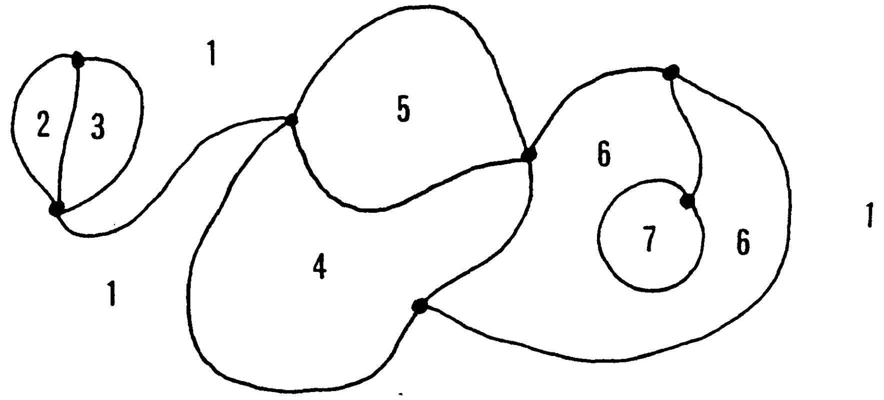
- * How can you fit a room 32 dm long?

- * Explore rooms with different lengths. Which ones can be fitted exactly with cupboards. Which cannot?

- * Suppose the factory decides to manufacture cupboards in 4 dm and 7 dm widths. Which rooms cannot be fitted now?

- * Investigate the situation for other pairs of cupboard sizes.
  Can you *predict* which rooms can or cannot be fitted?

NETWORKS

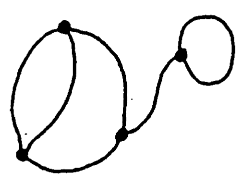


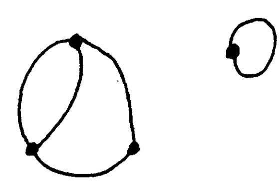
A network is a set of lines (or "arcs"), junctions (or "nodes") and spaces (or "regions") which compose a shape.

The network shown above is composed of 12 arcs, 7 nodes (marked with blobs) and 7 regions (these are numbered—notice that we have included the outside as a region).

Networks can be of two kinds:

Connected, like this . . . or disconnected like this . . .





Draw your own connected networks. Find a rule connecting the number of arcs, nodes and regions. Try to explain why your rule works.

Can you adapt your rule to work for disconnected networks?

A cube has 6 faces, 8 corners (or vertices) and 12 edges.
Explore the relationship between the number of faces, vertices and edges for other solid shapes.

Can you find any exceptional cases?

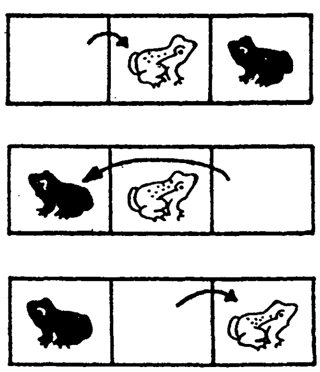
FROGS

These two frogs can change places in three moves

Move 1

Move 2

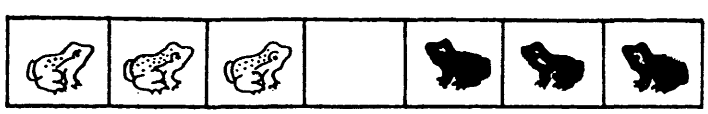
Move 3



Rules

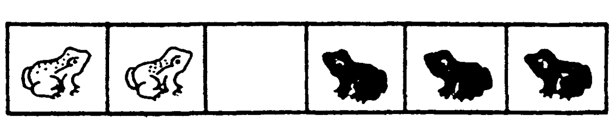
* A frog can *either* hop onto an adjacent square, *or* jump over one other frog to the vacant square immediately beyond it.
* The white frogs can only move from left to right the black frogs can only move from right to left.

The frogs shown below can be interchanged in 15 moves. Explain how.



How many moves would be needed to interchange 20 white and 20 black frogs?
– *n* white and *n* black frogs?

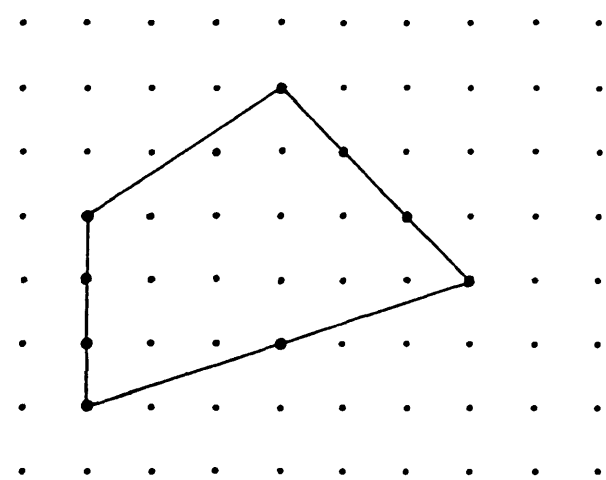
Now suppose that there are an *unequal* number of black and white frogs.
These frogs can be interchanged in 11 moves. Explain how.



How many moves are needed to interchange 15 white and 20 black frogs?
– *n* white and *m* black frogs?

DOTS

You will need a supply of dotty paper.



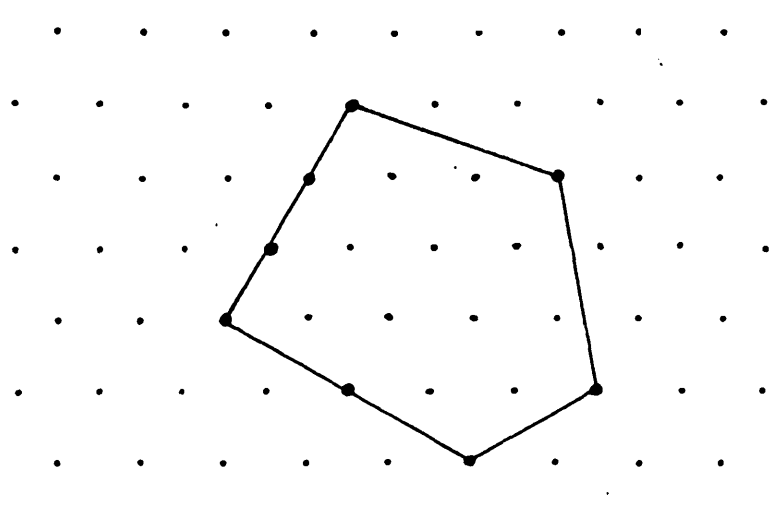
The quadrilateral shown in this diagram has an area of 16½ square units.

The perimeter of the quadrilateral passes through 9 dots.

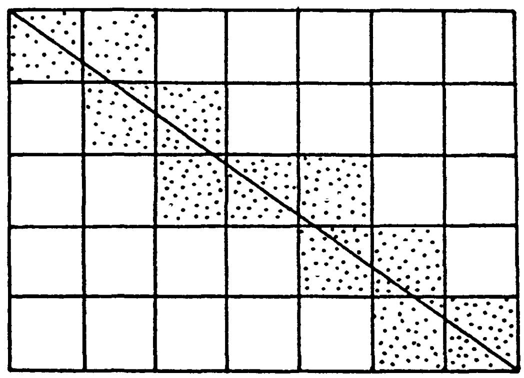
13 dots are contained within the quadrilateral.

Now draw your own shapes and try to find a relationship between the area, the number of dots on the perimeter and the number of dots inside each shape.

Try to find a similar result for a triangular dot lattice.
(You will of course have to redefine your unit of area).



DIAGONALS



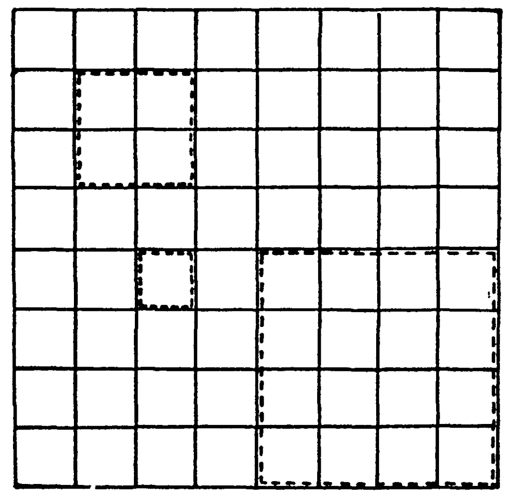
A diagonal of this 5×7 rectangle passes through 11 squares.

These have been shaded in the diagram.

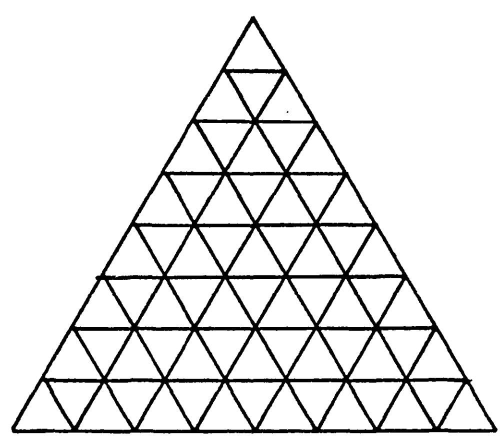
* Can you find a way of forecasting the number of squares passed through if you know the dimensions of the rectangle?

* How many squares will the diagonal of a 1000×800 rectangle pass through?

THE CHESSBOARD

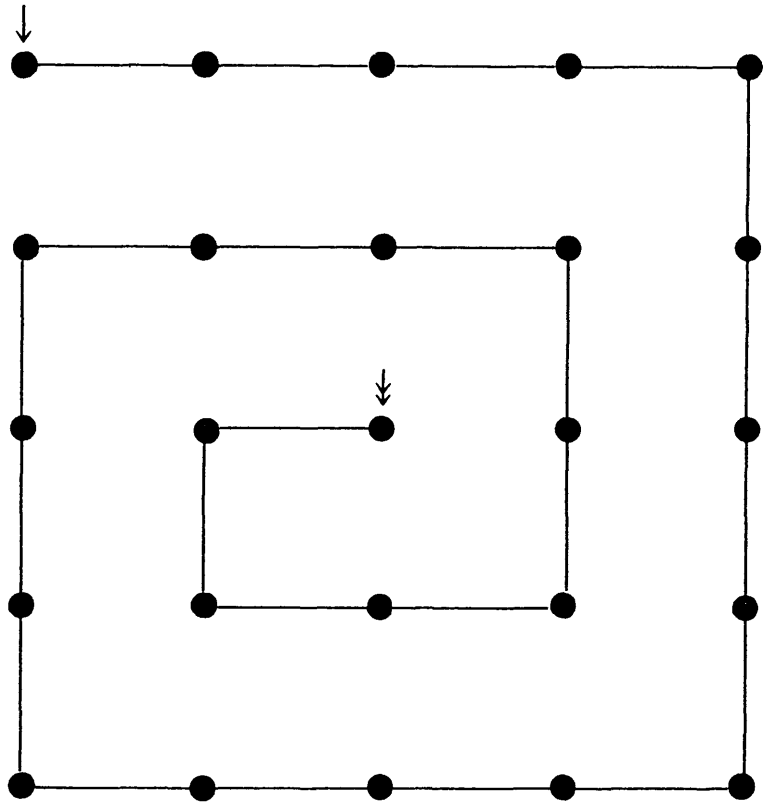


* How many squares are there on an 8×8 chessboard?
  (Three possible squares are shown by dotted lines).
* How many rectangles are there on the chessboard?
* Can you generalise your results for an $n \times n$ square?



* How many triangles are there on this 8×8 grid?
  How many point upwards?
  How many point downwards?
* Look for other shapes in this grid and count them.

THE SPIRAL GAME

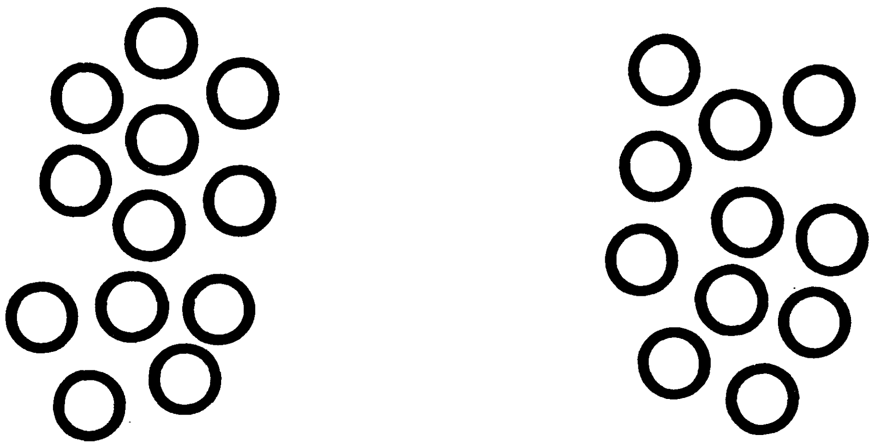


This is a game for two players. Place a counter on the dot marked "↓". Now take it in turns to move the counter between 1 and 6 dots along the spiral, always inwards. The first player to reach the dot marked "↡" wins.

Try to find a winning strategy.

Change the rule for moving in some way and investigate winning strategies.

NIM



This is a game for 2 players.

Arrange a pile of counters arbitrarily into 2 heaps.

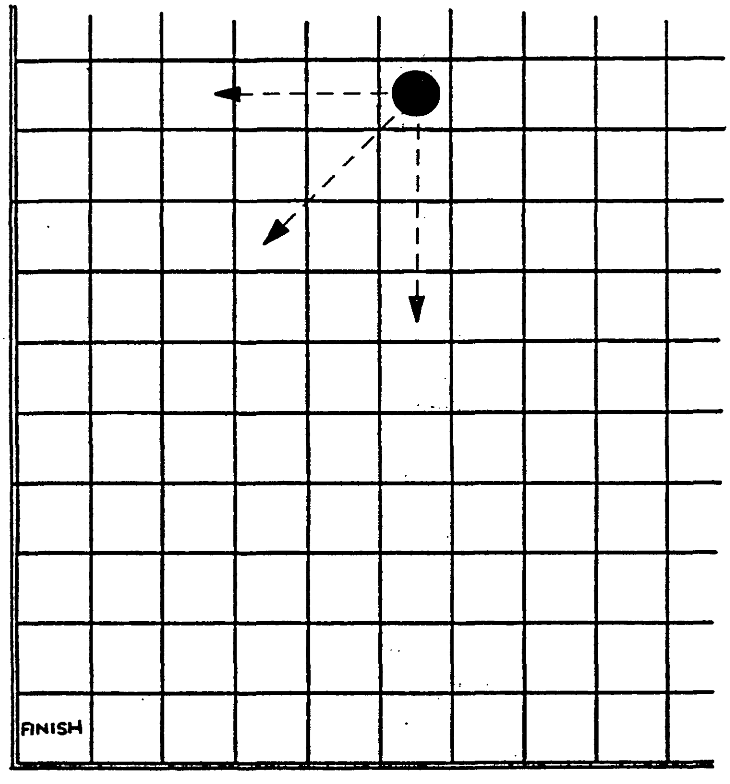
Each player in turn can remove as many counters as he likes from *one* of the heaps. He can, if he wishes, remove all the counters in a heap, but he must take at least one.

The winner is the player who takes the last counter.

Try to find a winning strategy.

Now change the game in some way and analyse your own version.

"FIRST ONE HOME"



This game is for two players. You will need to draw a large grid like the one shown, for a playing area.

Place a counter on any square of your grid.

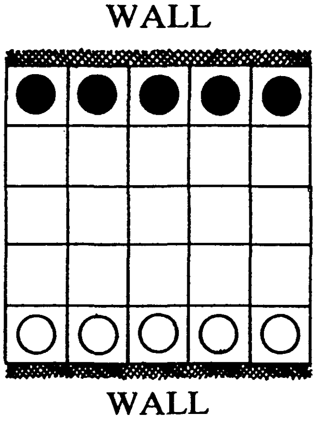
Now take it in turns to slide the counter *any number of squares* due West, South or Southwest, (as shown by the dotted arrows).

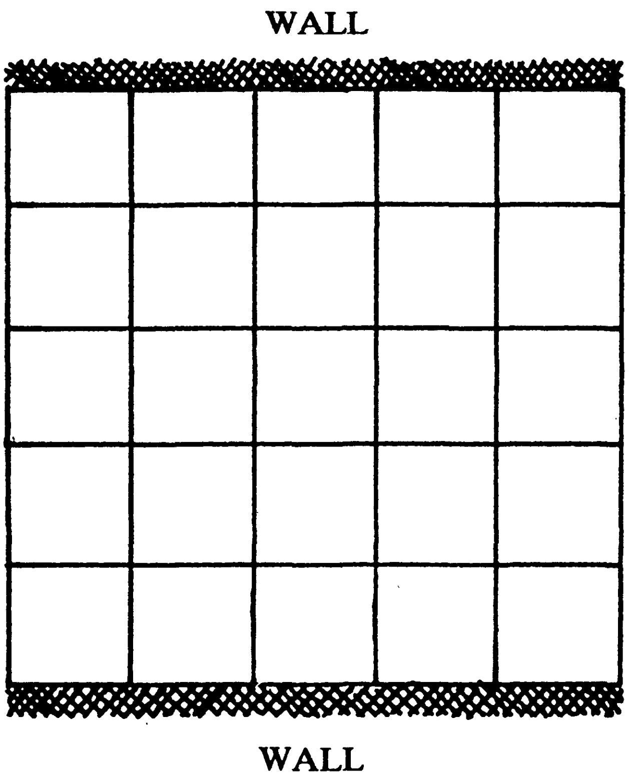
The first player to reach the square marked "Finish" is the winner.

PIN THEM DOWN!

A game for 2 players.

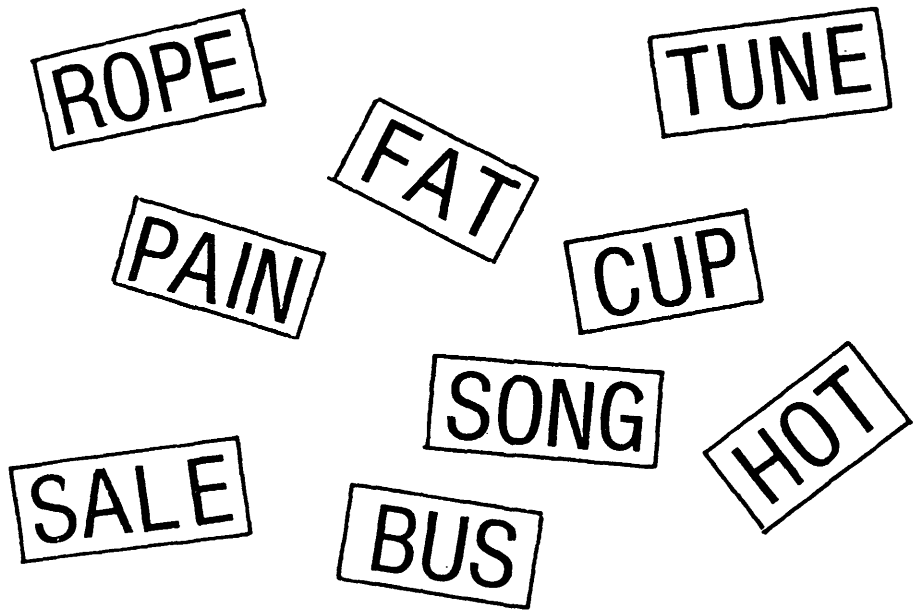
Each player puts counters of his colour in an end row of the board. The players take it in turns to slide one of their counters up or down the board *any* number of spaces.
No jumping is allowed. The aim is to prevent your opponent from being able to move by pinning him to the wall.





Can you find a winning strategy?

THE "HOT FAT TUNE" GAME



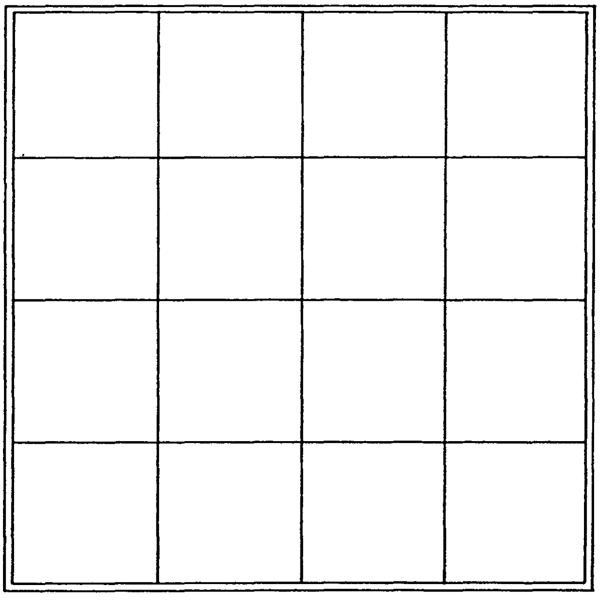
This is a game for two players.

Take it in turns to remove any one of the nine cards shown above.

The first player to hold three cards which contain the same letter is the winner.

Try to find a winning strategy.

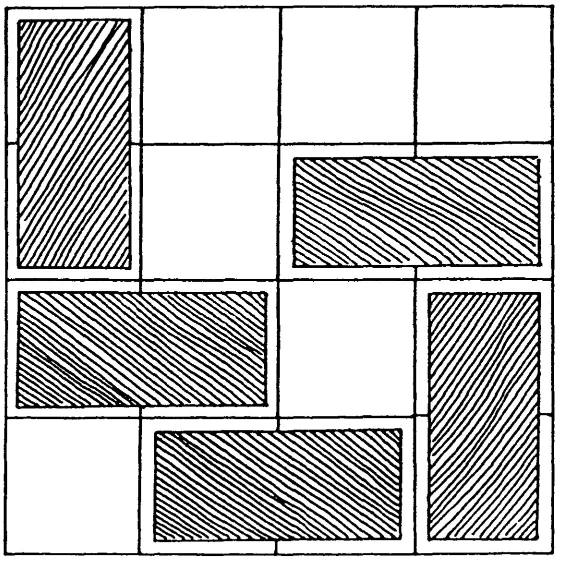
DOMINO SQUARE



This is a game for 2 players.
You will need a supply of 8 dominoes or 8 paper rectangles.

Each player, in turn, places a domino on the square grid, so that it covers two horizontally or vertically adjacent squares.
After a domino has been placed, it cannot be moved.
The last player to be able to place a domino on the grid wins the game.

For example, this board shows the first five moves in one game:

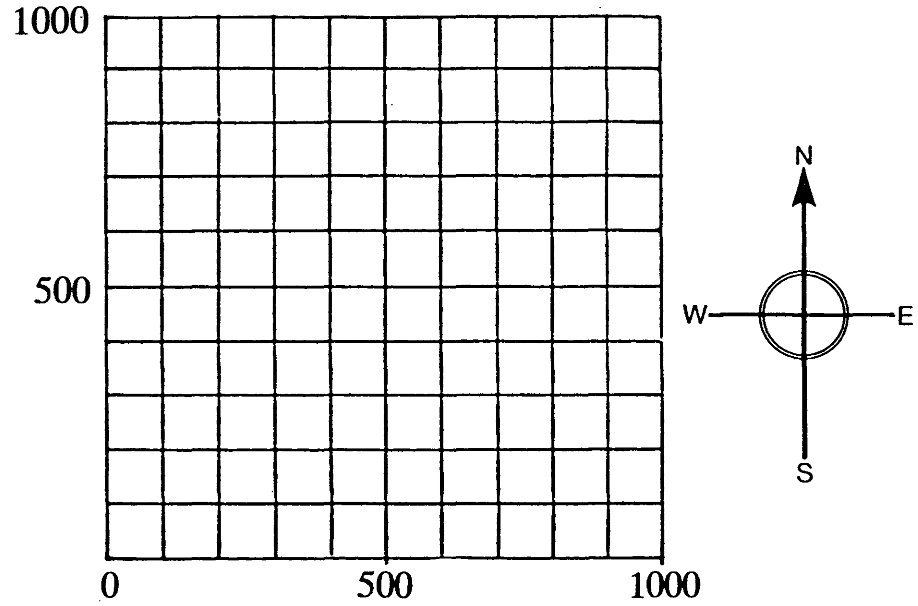


(It is player 2's turn. How can he win with his next move?)

Try to find a winning strategy.

THE TREASURE HUNT

This is a game for two players.
You will need a sheet of graph paper on which a grid has been drawn, like the one below. This grid represents a desert island.



One player "buries" treasure on this island by secretly writing down a pair of coordinates which describes its position.
For example, he could bury the treasure at (810,620).

The second player must now try to discover the exact location of the treasure by "digging holes", at various positions.
For example, she may say "I dig a hole at (200,200)".

The first player must now try to direct her to the treasure by giving clues, which can only take the form:
"Go North", "Go South", "Go East", "Go West", or "Go South and East" etc.
In our example, the first player would say "Go North and East".

- Take it in turns to hide the treasure.
- Play several games and decide who is the best "treasure hunter".
- How should the second player organise her "hole digging" in order to discover the treasure as quickly as possible?
- What is the *least* number of holes that need to be dug in order to be *sure* of finding the treasure, wherever it is hidden?

MARKING RECORD FORM

	Marker 1				Marker 2				Marker 3				Marker 4			
Script	R_0	R_1	M_1	M_2	R_0	R_1	M_1	M_2	R_0	R_1	M_1	M_2	R_0	R_1	M_1	M_2
A Emma																
B Mark																
C Ian																
D Colin																
E Peter																
F Paul																

Key:

Impression rank order	R_0
Raw mark	M_1
Mark rank order	R_1
Revised mark (if any)	M_2

NOTES ON MARKED SCRIPTS

Script A

Emma In part (iii) Emma was awarded only 1 mark out of 2 since her answer did not explain clearly that she had added the numbers from 1 to 11.

In part (iv) she was given 1 mark out of 2 as her answer showed evidence of a systematic approach although it was incomplete.

Script B

Mark In part (i) Mark's answer was correct and although no working was shown he was given both marks.

Although Mark's diagram for part (ii) is correct, there are three errors in his solution. He should have had 66 cubes$\times$4+12 and, in addition, his calculation of 45$\times$4+11 is incorrect. He was given 1 mark out of 4.

Script C

Ian Ian has misunderstood the question and assumed the tower to have a hollow middle.

In part (i) his answer is therefore wrong and he gets no marks.

In part (ii) he has made two errors: he assumed the tower has a hollow middle and has 13 layers. He was therefore given 2 marks out of 4.

In part (iii), his explanation of his calculation is not complete and so he scores 1 mark out of 2.

In part (iv) his answer is not correct and scores no marks.

Script D

Colin In part (ii) Colin has made two errors in multiplication for $h=11$ and $h=12$. Since each answer has been worked out independently using $c=h\times w$ only the error in $h=12$ need be penalised. So Colin scores 3 marks out of 4.

In part (iii) he scored both marks for a clear, complete and correct explanation of his method.

In part (iv) the three formulae on the left hand side are correct and sufficient to solve the problem, although they are not organised systematically. He was therefore awarded 1 mark out of 2.

Script E

Peter In part (ii) there is some doubt as to how Peter has worked out his answer. It may be that he has attempted to build onto the original tower and calculated the number of extra cubes needed but has forgotten to add on the 66. We are giving him the benefit of the doubt by taking this view although this may mean a slightly inflated mark. He was awarded 3 marks out of 4 for part (ii).

In part (iii) his explanation of his method is not very clear and he was awarded 1 mark out of 2.

Script F

Paul Paul's answer is of a very high standard. He was awarded 10 marks out of 10 despite the algebraic error in the last part.

triangles then use the formula of triangular numbers to one part of the tower.

$$\therefore \frac{x^2+x}{2} = \frac{11^2+11}{2} = \frac{132}{2} = 66$$

Now I multiply this number by 4.

$66 \times 4 = 264$

Now add twelve =

$264 + 12 = 276$ blocks 4

2

4. For a tower n cubes high:

Take away the middle n blocks; you are left with 4 blocks of n-1 high Then Use the following equation:

$$\frac{x^2+x}{2} = \frac{(n-1)^2+(n-1)}{2} =$$

Now multiply this number by 4.

$$\frac{(4n-1)^2+(4n-1)}{2}$$

Then add n to the total. 2

$$\frac{(4n-1)^2+(4n-1)}{2} + n =$$ NUMBER OF BLOCKS NEEDED TO MAKE A SKELETON TOWER OF n BLOCKS HIGH.

10

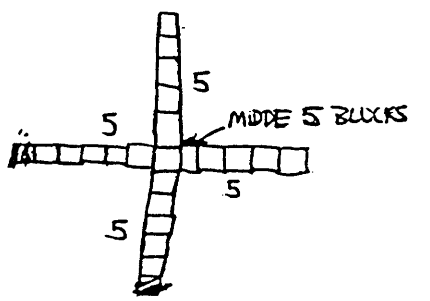
SCRIPT F PAUL

SKELETON TOWER.

The tower has 5 stories each going downwards in numerical order ie 1.2.3.4.5.
the formula for numerical numbers in order or triangular numbers is

$$\frac{x^2+x}{2}$$

The tower could be made up of two triangles each 11 blocks on the bottom side and ascend 11.9.7.5.3.1. The middle 5 blocks of one triangle are replaced by the other triangle whose middle 5 blocks has been removed. An overhead view of the triangles is in the shape of a cross.



1. For each part of the tower ie the five blocked base, use the formula for triangular numbers.

$$\frac{x^2+x}{2} = \frac{5^2+5}{2} = \frac{30}{2} = 15$$

now multiply this by 4 (the four parts)

$15 \times 4 = 60$

Now add 6 (the middle six blocks)

$60 + 6 = 66$ Blocks.

2

2 & 3 To build a tower 12 cubes high

Take away the middle twelve blocks so you are left with four eleven blocked

continued

SCRIPT E PETER

(1) 15 x 4 = 60 + 6 = 66 . Cubes are needed to build this tower. **2**

(2)
4 x 6 = 24
4 x 7 = 28
4 x 8 = 32
4 x 9 = 36
4 x 10 = 40
4 x 11 = 44
+ 6 + 6
210 **3**

210 cubes are needed to make a tower 12 cubes high

3) I counted the number of cubes down one side on the top edge which came to 5 Excluding the centre one. Adding one on each time to 5 and the multiplying by 4 each time then adding the 6 centre ones I came to the answer of 210 cubes. **1**

(6)

SCRIPT D COLIN

Skeleton Tower:

1. The amount of cubes needed to build the tower are 66 **2**

2.

Height (H)	amount of cubes (c)	width of base (w)	amount of steps (s)
6 ~~7~~	66	11	5
7 ~~8~~	~~[illegible]~~ 91	13	6
8 ~~9~~	~~2~~120	15	7
9 ~~10~~	153	17	8
10 ~~11~~	190	19	9
11 ~~12~~	221	21	10
12	256	23	11

3

Amount of steps = height minus one.
Width of base = amount of steps multiplied by two add one **2**
Amount of cubes = height multiplied by width of base

So a tower of height 12 would need 256 cubes to be constructed out of.

$w = 2s + 1$ $\quad s = \frac{c}{w} - 1$

$c = H \times w$ $\quad w = \frac{c}{H}$ $\quad H = \frac{c}{w}$ **1**

$S = H - 1$ $\quad H = s + 1$

As the height increases by one the width goes up by two

(8)

SCRIPT C IAN

Skeleton Tower

1) ~~20+16+8+4+1 = 58 cubes~~. 20+16+12+8+4+1 = 61 **0**

2) ~~1+4+8+12+16+20~~ 1+4+8+12+16+20+24+28+32+36+40+44+48

= 313 cubes. 85 113 135 171 211 255 20815 **2**

3) In question 2 there is a pattern which goes up in 4,s eg 1st level = 1, 2nd = 4, 3rd = 8 and so on, so I just did that until I came to the height of 12 cubes. **1**

4) cubes = ~~height of cubes~~ 1 layer x4 xn

A quarter of the ~~base~~ bottom layer is 5 in Q1, multiply 5x4, and you have the number of cubes in the bottom layer. 20 mins **0**

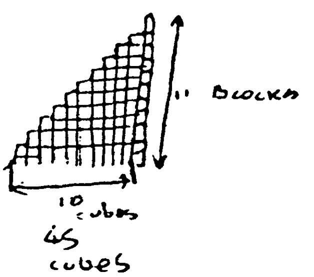
(3)

SCRIPT B   MARK

SKELETON TOWER

b) (1) 66 cubes. 2

(2) 192 cubes.

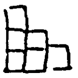


45 cubes ×4 + 11 = 192

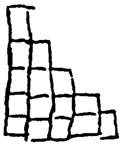
1

(3)

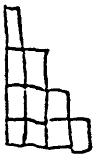
a tower 4 high:-

 = 6 x 4 + 4 = 28

" " 6 high:-

 15 x 4 = 60 + 6 = 66

" " 5 high:-

 10 x 4 = 40 + 5 = 45.

" " 3 high  3 x 4 = 12 + 3 = 15

" " 2  1 x 4 = 4 + 2 = 6

" " 1 0 x 4 = 0 + 1 = 1

⇓

a table will help me to spot patterns

height of towers	1	2	3	4	5	6	12
no. of blocks used.	1	6	15	28	45	66	276

difference pattern 5 9 13 17 21

4 4 4 4

The difference pattern of the difference pattern = 4.

height of tower times, height of tower no of blocks used = no. of blocks used [is a pattern]

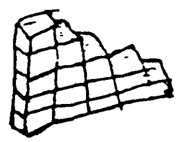
1

8

SCRIPT A EMMA

SKELETON TOWER

1) Not including the central perpendicular column each 'Side Part'

eg 

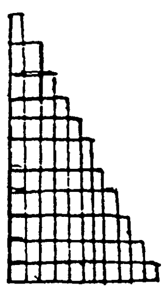
consists of 15 cubes ∴ 15 x 4 = 60.

60 + 6 = 66 This is the no. of cubes needed 2

The 5 up above is the central column.

2)

Each arm would consist of:-

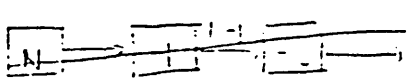


66 x 4 = 264

264 + 12 = 276 blocks. would equal a tower which is 12 cubes high. 4

3) I just worked out the no. of cubes for 1 arm by starting at 11 cubes high and decreasing down to 1. I multiplied this by 4 as there are 4 arms, I then added the total height of the tower on to this result. 1

4). As each arm starts 1 cube down I would firstly write



I have decided to try some simpler examples to see if I can spot some patterns

continued

triangles then use the formula of triangular numbers for one part of the tower.

$$\therefore \frac{x^2+x}{2} = \frac{11^2+11}{2} = \frac{132}{2} = 66$$

Now I multiply this number by 4.

$66 \times 4 = 264$

Now add twelve =

$264 + 12 = 276$ blocks

4. For a tower n cubes high:

Take away the middle n blocks; you are left with 4 blocks of $n-1$ high. Then use the following equation

$$\frac{x^2+x}{2} = \frac{(n-1)^2+(n-1)}{2} \; [illegible] =$$

Now multiply this number by 4.

$$\frac{(4n-1)^2+(4n-1)}{2}$$

Then add n to the total.

$$\frac{(4n-1)^2+(4n-1)}{2} + n$$ = NUMBER OF BLOCKS NEEDED TO MAKE A SKELETON TOWER OF n BLOCKS HIGH.

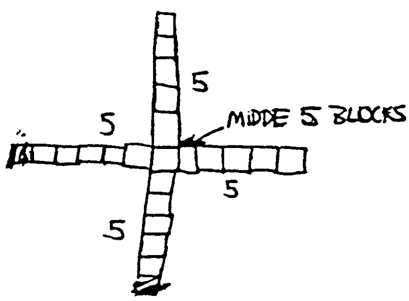
SCRIPT F PAUL

SKELETON TOWER.

The tower has 5 stories each going ~~upwards~~ downwards in numerical order ie 1.2.3.4.5.
the formular for numerical numbers in order or triangular numbers is

$$\frac{x^2+x}{2}$$

The tower could be made up of two triangles each 11 blocks on the bottom side and assend 11.9.7.5.3.1. The middle 5 blocks of one triangle are replaced by the other triangle whose middle 5 blocks has been removed. An overhead view of the triangles is in the shape of a cross.



1. For each part of the tower ie the five blocked base, use the formula for triangular numbers.

$$\frac{x^2+x}{2} = \frac{5^2+5}{2} = \frac{30}{2} = 15$$

now multiply this by 4 (the four parts)

15 x 4 = 60

Now add 6 (the middle six blocks)

60 + 6 = 66 Blocks.

2 & 3 To build a tower n cubes high

Take away the middle twelve blocks so you are left with four eleven blocked

continued

SCRIPT E PETER

(1) 15 x 4 = 60 + 6 = 66 Cubes are needed to build this tower.

(2)

4 x 6	=	24
4 x 7	=	28
4 x 8	=	32
4 x 9	=	36
4 x 10	=	40
4 x 11	=	44
+ 6	+	6
		210

210 cubes are needed to make a tower 12 cubes high

3) I counted the number of cubes down one side on the top edge which came to 5 Excluding the centre one. Adding an on each time to 5 and the multiplying by 4 each time then adding the 6 centre ones I came to the answer of 210 cubes.

Skeleton Tower:

1. The amount of cubes needed to build the tower are 66

2. Height (H)	amount of cubes (c)	width of base (w)	amount of steps (s)
6 ~~7~~	66	11	5
7 ~~8~~	~~80~~ 91	13	6
8 ~~9~~	120	15	7
9 ~~10~~	153	17	8
10 ~~11~~	190	19	9
11 ~~12~~	221	21	10
12	256	23	11

Amount of steps = height minus one.
Width of base = amount of steps multiplied by two add one
Amount of cubes = height multiplied by width of base

So a tower of height 12 would need 256 cubes to be constructed out of.

$w = 2s + 1$ $\quad s = \frac{c}{w} - 1$

$c = H \times w$ $\quad w = \frac{c}{H}$ $\quad H = \frac{c}{w}$

$s = H - 1$ $\quad H = s + 1$

As the height increases by one the width goes up by two.

SCRIPT C IAN

Skeleton Tower

1) ~~[illegible]~~ 20+16+12+8+4+1 = 61

2) ~~[illegible]~~ 1+4+8+12+16+20+24+28+32+36+40+44+48
= 313 cubes.

85 113 135 171 211 255 2?8+5

3) In question 2 there is a pattern which goes up in 4,s eg 1st level = 1, 2nd = 4, 3rd = 8 and so on, so I just did that until I came to the height of 12 cubes.

4) cubes = ~~[illegible]~~ 1 layer × 4 × n

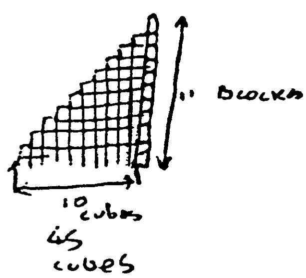
A quarter of the ~~[illegible]~~ bottom layer is 5 in Q1, multiply 5×4, and you have the number of cubes in the bottom layer. 20 mins

SCRIPT B MARK

SKELETON TOWER

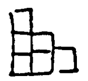
b) (1) 66 cubes.

(2) 192 cubes.

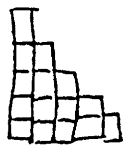


45 cubes ×4 + 11 = 192

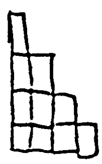
a tower 4 high:-

 = 6 x 4 + 4 = 28

" " 6 high:-

 15 x 4 = 60 + 6 = 66

" " 5 high:-

 10 x 4 = 40 + 5 = 45.

" " 3 high  3 x 4 = 12 + 3 = 15

" " 2  1 x 4 = 4 + 2 = 6

" " 1 0 x 4 = 0 + 1 = 1

⇓

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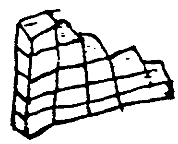
eg 

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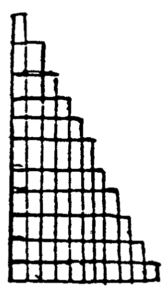

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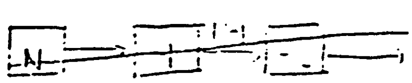


I have decided to try some simpler examples to see if I can spot some patterns

continued

SKELETON TOWER . . . MARKING SCHEME

(i) **Showing an understanding of the problem by dealing correctly with a simple case.**

Answer: 66

2 marks for a correct answer (with or without working).

Part mark: Give 1 mark if a correct method is used but there is an arithmetical error.

(ii) **Showing a systematic attack in the extension to a more difficult case.**

Answer: 276

4 marks if a correct method is used and the correct answer is obtained.

Part marks: Give 3 marks if a correct method is used but the work contains an arithmetical error or shows a misunderstanding (e.g. 13 cubes in the centre column).

Give 2 marks if a correct method is used but the work contains two arithmetical errors/misunderstandings.

Give 1 mark if the candidate has made some progress but the work contains more than two arithmetical errors/misunderstandings.

(iii) **Describing the methods used.**

2 marks for a correct, clear, complete description of what has been done providing more than one step is involved.

Part mark: Give 1 mark if the description is incomplete or unclear but apparently correct.

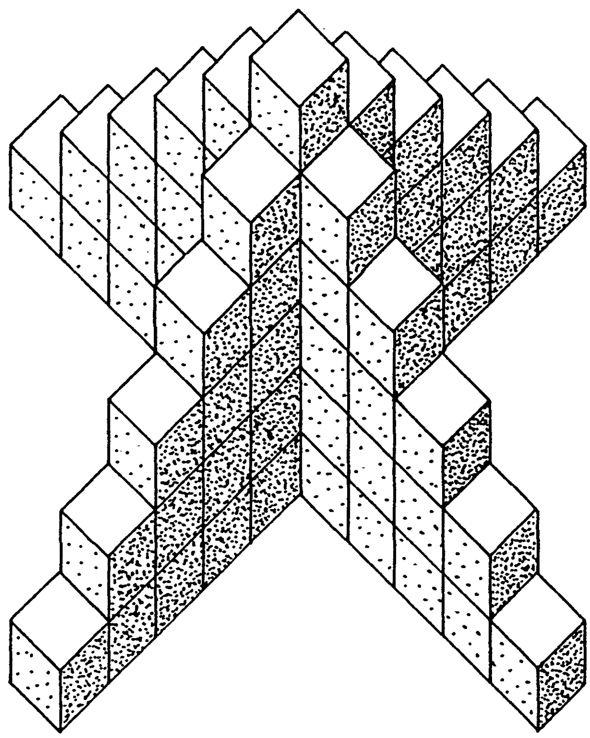
(iv) **Formulating a general rule verbally or algebraically.**

2 marks for a correct, clear, complete description of method.

Accept "number of cubes$=n(2n-1)$" or equivalent for 2 marks. Ignore any errors in algebra if the description is otherwise correct, clear and complete.

Part mark: Give 1 mark if the description is incomplete or unclear but shows that the candidate has some idea how to obtain the result for any given value of n.

SKELETON TOWER



(i) How many cubes are needed to build this tower?

(ii) How many cubes are needed to build a tower like this, but 12 cubes high?

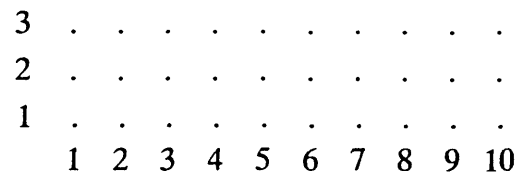
(iii) Explain how you worked out your answer to part (ii).

(iv) How would you calculate the number of cubes needed for a tower *n* cubes high?

A TREASURE HUNT PROBLEM

This is a game for two players.

The diagram below represents an island, and each dot represents a possible location for some buried treasure. (In this case there are 30 possible hiding places).



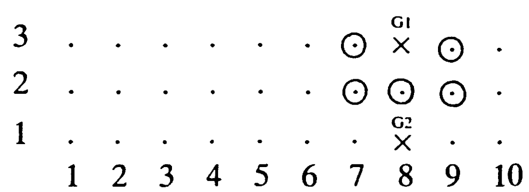
One player has to guess the location of the treasure, and the other has to provide a "clue" after each guess, which can only be of the following kind:

After the first guess, the clue is either "warm" or "cold" according to whether the treasure is located at a neighbouring point or not.

After each succeeding guess, the clue is either "warmer", "colder", or "same temperature", depending on whether the guess is closer to, further away from or the same distance from the treasure as the previous guess.

The aim is to discover the treasure with as few guesses as possible.

* In the sample game shown below, the first guess, G1, was (8,3). The clue given was "cold", so the treasure is not on any neighbouring points (shown with a ⊙).



The second guess, G2, was (8,1) . . .
Show that, wherever it is buried, the treasure can always be located with a total of 5 guesses (including G1 and G2). Is this the minimum number?

* Now try to find the minimum number of guesses needed for a different grid . . .

* What is the best "guessing" strategy?

Support Materials